

# OBSERVABILITY AND NONLINEAR FILTERING

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**ABSTRACT.** This paper develops a connection between the asymptotic stability of nonlinear filters and a notion of observability. We consider a general class of hidden Markov models in continuous time with compact signal state space, and call such a model observable if no two initial measures of the signal process give rise to the same law of the observation process. We demonstrate that observability implies stability of the filter, i.e., the filtered estimates become insensitive to the initial measure at large times. For the special case where the signal is a finite-state Markov process and the observations are of the white noise type, a complete (necessary and sufficient) characterization of filter stability is obtained in terms of a slightly weaker detectability condition. In addition to observability, the role of controllability in filter stability is explored. Finally, the results are partially extended to non-compact signal state spaces.

## 1. INTRODUCTION

Consider the deterministic linear control system

$$\begin{aligned}\frac{d}{dt}x(t) &= Ax(t) + \Sigma u(t), \\ y(t) &= Cx(t),\end{aligned}$$

where  $x(t)$  is the system state,  $u(t)$  is the control input and  $y(t)$  is the observation signal. Such a system is called *observable* if there exist no  $x \neq x'$  such that  $\{y(t) : t \geq 0\}$  is the same when  $x(0) = x$  and  $x(0) = x'$  (for any control  $u(t)$ ), and is called *controllable* if for any  $x, x'$  and  $t > 0$ , there is a control signal  $u(t)$  such that the solution with  $x(0) = x$  satisfies  $x(t) = x'$ . It is well known [6, 26] that observability and controllability are intimately related with the asymptotic properties of the conditional estimates in the linear filtering problem

$$\begin{aligned}dX_t &= AX_t dt + \Sigma dW_t, \\ dY_t &= CX_t dt + dB_t,\end{aligned}$$

where  $W_t$  and  $B_t$  are independent standard Wiener processes. In particular, it is found that the filtered estimates become insensitive to the law of  $X_0$  at large times, i.e.,<sup>1</sup>  $|\mathbf{E}^\mu(f(X_t)|\mathcal{F}_t^Y) - \mathbf{E}^\nu(f(X_t)|\mathcal{F}_t^Y)| \rightarrow 0$  as  $t \rightarrow \infty$  for any pair of initial laws  $X_0 \sim \mu, \nu$ , whenever the associated linear control system is observable and controllable [26]. This is called the *stability* property of the filtering problem, and is of significant importance from the practical point of view as it ensures the robustness

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<sup>1</sup> Here  $\mathbf{P}^\mu$  is the law under which  $X_0 \sim \mu$ ,  $\mathcal{F}_t^Y = \sigma\{Y_s : s \leq t\}$ , and the versions of the conditional expectations are chosen to coincide with those computed by the Kalman filter.

of the filtered estimates with respect to modelling errors and approximations. The purpose of this paper is to demonstrate that the connection between observability, controllability, and the linear filtering problem has a natural counterpart in a large variety of nonlinear filtering problems for Markovian signal-observation models.

We consider a signal process  $X_t$  and observation process  $Y_t$  in continuous time, where both  $X_t$  and  $(X_t, Y_t)$  are assumed to be Markov processes and  $Y_t$  is assumed to satisfy a mild condition which ensures, in essence, that the observation noise is memoryless (the most common observation models in continuous time, additive white noise observations and counting observations, both satisfy this requirement). In addition, we will mostly assume that the signal process takes values in a compact state space. The significance of the compactness assumption and some extensions of the results to the non-compact case are discussed in section 8.

In this general setting, the model is called *observable* if there do not exist initial measures  $\mu \neq \nu$  such that  $\{Y_t : t \geq 0\}$  has the same law when  $X_0 \sim \mu$  and  $X_0 \sim \nu$ . One of the main results of this paper (corollary 4.6) is that if the model is observable, then  $|\mathbf{E}^\mu(f(X_t)|\mathcal{F}_t^Y) - \mathbf{E}^\nu(f(X_t)|\mathcal{F}_t^Y)| \rightarrow 0$  as  $t \rightarrow \infty$   $\mathbf{P}^\mu$ -a.s. for any bounded continuous  $f$  and any pair of absolutely continuous initial laws  $\mu \ll \nu$ .

On the other hand, the notion of *controllability* is replaced by a certain regularity property of the signal transition probabilities (see definition 7.8). Under the additional assumption that the observations are of the white noise type and nondegenerate, we show that observability and regularity of the signal imply that  $|\mathbf{E}^\mu(f(X_t)|\mathcal{F}_t^Y) - \mathbf{E}^\nu(f(X_t)|\mathcal{F}_t^Y)| \rightarrow 0$  as  $t \rightarrow \infty$   $\mathbf{P}^\mu$ -a.s. for any bounded continuous  $f$  and initial laws  $\mu, \nu$  (corollary 7.11), without the absolute continuity requirement. When the signal is the solution of a stochastic differential equation, then the regularity property is directly related to the controllability of an associated deterministic control system. This result is thus entirely parallel to the observability-controllability criterion for the stability of the Kalman filter.

A natural test case for the general theory is the setting where the signal state space is a finite set. The simplicity of this setting allows a particularly transparent insight into the nature of the observability and controllability properties, while this setting is also of particular practical importance due to the fact that the associated nonlinear filters are finite dimensionally computable. By combining the general results in this paper with known filter stability results for ergodic signals [4], we find that a complete characterization of filter stability is possible for finite state signals with nondegenerate white noise type observations. In particular, we will find necessary and sufficient criteria for stability (theorems 6.12 and 7.2) which can be verified explicitly for a given model through straightforward linear algebra techniques. The fact that such a complete characterization is possible, albeit in a particularly simple case, suggests that the notion of observability used in this paper is in some sense a fundamental ingredient of the filter stability problem.

The stability of nonlinear filters has been studied actively in the last few years following the pioneering contributions of Ocone and Pardoux [26] and Zeitouni *et al.* [13, 1]. An excellent overview of previous work and an extensive list of references can be found in [7]. The majority of the results on this topic assume that the signal is an ergodic Markov process and that the observation process is of the white noise type. Such results are complementary to the results obtained in this paper; indeed, for the complete characterization of stability in the finite state setting, it is essential to combine our results with results that are specific to ergodic signals.

Moreover, ergodic results may fail to hold true for certain degenerate observation models (see the counterexample in [4]), while the results of this paper still hold in that setting. On the other hand, much is known about the rate of convergence of differently initialized filters in particular cases (see, e.g., [13, 1, 4]), while our results do not provide such information. In a much more general setting, some results on the asymptotic properties of nonlinear filtering errors can be found in Clark, Ocone and Coumarbatch [9]. Their point of view is close to the one used in this paper, but their results do not establish stability of the filter.

The approach in this paper is inspired by the observation of Chigansky and Liptser [8] that, by virtue of a martingale convergence argument, certain predictive estimates of the observations are always stable. This implies, in particular, that the filtered estimates of the signal are stable for a particular class of functions, which can be written as the conditional mean of a functional of the observation process given the initial value of the signal (see section 4). The heart of the argument that leads to the observability criterion is the characterization of this class of functions (proposition 3.6), which is achieved using a corollary of the Hahn-Banach theorem.

The remainder of this paper is set up as follows. In section 2, we introduce the general signal-observation model that will be used in most of the paper, and fix the notation for the remainder of the paper. Section 3 is devoted to the study of the notion of observability and its connection to the class of functions that can be obtained as predictive estimates of the observations. Section 4 connects these concepts to the stability of the filter. In section 5, we show how the notion of observability can be characterized in the common cases of white noise type and counting observations, and we find a particularly simple sufficient criterion for observability in those cases. Section 6 is devoted to the finite state case; an explicit criterion is found for observability, and stability is completely characterized for absolutely continuous initial measures. Section 7 explores the connection between controllability, regularity, and the stability of filters for arbitrary initial conditions; a complete characterization is again given for the finite state case. Finally, section 8 discusses the significance of the compactness assumption made in the previous sections, and provides some partial extensions of previous results. A simple but apparently unknown result for the Kalman filter is briefly discussed in the appendix.

Before proceeding to the main part of the paper, we make a few general remarks.

*Remark 1.1.* The main results of this paper can be adapted in a straightforward fashion to the discrete time setting.

*Remark 1.2.* It is not difficult to show that when our general notions of observability and controllability are applied to the linear filtering model, one obtains precisely the classical observability-controllability criteria for the Kalman filter. Unfortunately, the results in section 8 for the non-compact case are not sufficiently powerful to recover the stability of the Kalman filter, except then the signal itself is asymptotically stable (i.e., the matrix  $A$  has only eigenvalues with strictly negative real parts). The latter case is not particularly interesting for the Kalman filter, as the more general detectability criterion (see [26] and the appendix) makes observability irrelevant in this setting. The fact that the Kalman filter with an unstable signal is not covered is a major shortcoming of the results in this paper.

*Remark 1.3.* With the exception of the Kalman filter, the finite state case, and observations with an invertible observation function (lemma 5.6), the observability

property appears to be difficult to verify for a given model. For practical applications, it is thus necessary to develop explicitly verifiable sufficient criteria for observability (see section 7.3 for further discussion).

## 2. THE SIGNAL-OBSERVATION MODEL

The goal of this section is to set up the model for the signal and observation processes, and to fix the notation that will be used in the following.

Let us begin by introducing the basic objects that make up the model.

- (1) The *signal state space*  $\mathbb{S}$  is a compact Polish space.
- (2) The *observation state space*  $\mathbb{O} = \mathbb{R}^p$  for some  $p < \infty$ .
- (3) The *signal-observation process*  $(X_t, Y_t)_{t \in [0, \infty[}$  is a time-homogeneous  $\mathbb{S} \times \mathbb{O}$ -valued Feller-Markov process with càdlàg paths.
- (4) The *signal process*  $(X_t)_{t \in [0, \infty[}$  is a Feller-Markov process in its own right.
- (5) The *observation process*  $(Y_t)_{t \in [0, \infty[}$  has conditionally independent increments given the signal process  $(X_t)_{t \in [0, \infty[}$ , and  $Y_0 = 0$ .

This can be viewed as a hidden Markov model in continuous time, where  $Y_t$  is the observable component and  $X_t$  is the nonobservable component.

For any locally compact Polish space  $S$ , we denote by  $\mathcal{B}(S)$  the Borel  $\sigma$ -algebra, by  $\mathcal{C}(S)$  the space of continuous functions, by  $\mathcal{C}_b(S)$  the space of bounded continuous functions, by  $\mathcal{C}_0(S)$  the space of continuous functions that vanish at infinity, by  $\mathcal{M}(S)$  the space of finite signed measures on  $\mathcal{B}(S)$ , by  $\mathcal{P}(S)$  the space of probability measures on  $\mathcal{B}(S)$ , and by  $\mathcal{M}_c(S)$  ( $\mathcal{P}_c(S)$ ) the finite signed (probability) measures with compact support. Note that when  $S$  is compact,  $\mathcal{C}(S) = \mathcal{C}_b(S) = \mathcal{C}_0(S)$ .

It is convenient to construct the signal-observation process  $(X_t, Y_t)_{t \in [0, \infty[}$  on its canonical probability space. To this end, define  $\Omega^X = D([0, \infty[; \mathbb{S})$  and  $\Omega^Y = D([0, \infty[; \mathbb{O})$ , i.e.,  $\Omega^X$  and  $\Omega^Y$  are the spaces of  $\mathbb{S}$ -valued and  $\mathbb{O}$ -valued càdlàg paths, endowed with the Skorokhod topology. We will work on the probability space  $\Omega = \Omega^X \times \Omega^Y$ , equipped with its Borel  $\sigma$ -algebra  $\mathcal{F} = \mathcal{B}(\Omega^X \times \Omega^Y)$ , and choose  $X_t : \Omega \rightarrow \mathbb{S}$  and  $Y_t : \Omega \rightarrow \mathbb{O}$  to be the canonical processes  $X_t(x, y) = x(t)$  and  $Y_t(x, y) = y(t)$ . Furthermore, we define the natural filtrations

$$\mathcal{F}_t^X = \sigma\{X_s : s \leq t\}, \quad \mathcal{F}_t^Y = \sigma\{Y_s : s \leq t\}, \quad \mathcal{F}_t = \sigma\{(X_s, Y_s) : s \leq t\},$$

and the filtration generated by the observation increments

$$\mathcal{G}_t^Y = \sigma\{Y_s - Y_0 : s \leq t\}.$$

We will denote  $\mathcal{F}^X = \mathcal{F}_\infty^X = \bigvee_{t \geq 0} \mathcal{F}_t^X$ , and we define  $\mathcal{F}^Y$  and  $\mathcal{G}^Y$  similarly.

Let  $T_t : \mathcal{C}_0(\mathbb{S} \times \mathbb{O}) \rightarrow \mathcal{C}_0(\mathbb{S} \times \mathbb{O})$  and  $P_t(x, y, A)$  ( $t \in [0, \infty[$ ,  $A \in \mathcal{B}(\mathbb{S} \times \mathbb{O})$ ) be the Markov semigroup of the signal-observation process and the associated transition probabilities. By the Feller assumption, we can construct a process with càdlàg paths which possesses the desired transition probabilities [19, theorem 17.15]. Hence there exists a family of probability measures  $\{\mathbf{P}_{(x,y)} : (x, y) \in \mathbb{S} \times \mathbb{O}\}$  on  $(\Omega, \mathcal{F})$  such that for every  $(x, y)$ , the process  $(X_t, Y_t)$  is a Markov process with respect to the filtration  $\mathcal{F}_t$  under  $\mathbf{P}_{(x,y)}$  with transition probabilities  $P_t(x, y, A)$  and initial law  $(X_0, Y_0) \sim \delta_{\{(x,y)\}}$ , and  $(x, y) \mapsto \mathbf{P}_{(x,y)}(A)$  is measurable for every  $A \in \mathcal{F}$ . In particular, under the probability measure

$$\mathbf{P}_\mu(A) = \int_{\mathbb{S} \times \mathbb{O}} \mathbf{P}_{(x,y)}(A) \mu(dx, dy), \quad A \in \mathcal{F}, \quad \mu \in \mathcal{P}(\mathbb{S} \times \mathbb{O}),$$

$(X_t, Y_t)$  is a Markov process with respect to the filtration  $\mathcal{F}_t$  with transition probabilities  $P_t(x, y, A)$  and initial law  $(X_0, Y_0) \sim \mu$ . We recall that the Markov property can be expressed as follows [29, proposition III.1.7]: for bounded  $\mathcal{F}$ -measurable  $\xi$

$$\mathbf{E}_\mu(\xi \circ \theta_t | \mathcal{F}_t) = \mathbf{E}_{(X_t, Y_t)}(\xi) \quad \mathbf{P}_\mu\text{-a.s.} \quad \text{for all } t > 0,$$

where  $\mathbf{E}_\mu$ ,  $\mathbf{E}_{(x, y)}$  denote the expectations with respect to the measures  $\mathbf{P}_\mu$  and  $\mathbf{P}_{(x, y)}$ , and  $\theta_t : \Omega \rightarrow \Omega$  is the canonical shift  $\theta_t(x, y)(s) = (x(s+t), y(s+t))$ .

It is convenient, without loss of generality, to replace the various  $\sigma$ -algebras and filtrations defined above by their usual augmentations with respect to the family  $\{\mathbf{P}_\mu : \mu \in \mathcal{P}(\mathbb{S} \times \mathbb{O})\}$  [29, section 1.4], and we will make this replacement from this point onwards. A significant advantage of this choice is that if a bounded process  $Z_t$  has càdlàg paths, and the filtration  $\mathcal{G}_t$  satisfies the usual conditions, then we can choose a version of  $\mathbf{E}(Z_t | \mathcal{G}_t)$ , for every time  $t$ , so that the process  $t \mapsto \mathbf{E}(Z_t | \mathcal{G}_t)$  has càdlàg paths [12, chapter VI, theorem 47], [28, theorem 6]. *In the following, whenever such processes are encountered, their càdlàg versions are always implied.*

Finally, let us make precise the conditions on the signal and observations, i.e., that the signal is a Markov process in its own right and that the observation process has conditionally independent increments given the signal process. Both these properties can be simultaneously introduced through the following requirement.

- The signal is a Markov process in its own right, and the observation process has conditionally independent increments given the signal process, in the following sense: if the random variable  $\xi$  is bounded and  $\mathcal{F}^X \vee \mathcal{G}^Y$ -measurable, then the map  $(x, y) \mapsto \mathbf{E}_{(x, y)}(\xi)$  does not depend on  $y$ .

Using the Markov property of  $(X_t, Y_t)$ , this implies that  $\mathbf{E}_\mu(\xi | \mathcal{F}_s) = \mathbf{E}_\mu(\xi | X_s)$   $\mathbf{P}_\mu$ -a.s. whenever  $\xi$  is  $\sigma\{X_t : t \geq s\}$ -measurable, which establishes that  $X_t$  is an  $\mathcal{F}_t$ -Markov process as desired. On the other hand, we find that for any bounded,  $\sigma\{Y_{t+s} - Y_s : t > 0\}$ -measurable random variable  $\xi$ , there exists a measurable function  $f : \mathbb{S} \rightarrow \mathbb{R}$  such that  $\mathbf{E}_\mu(\xi | \mathcal{F}_s) = f(X_s)$   $\mathbf{P}_\mu$ -a.s. for any initial measure  $\mu$  (by the Markov property). This expresses the fact that the additional randomness introduced by the observation process is memoryless. As we will see, the two most common types of observations encountered in continuous time problems, white noise type observations and counting observations, satisfy this property.

It remains to note that the assumption  $Y_0 = 0$  means that we will be interested in initial measures of the form  $\mu \times \delta_{\{0\}} \in \mathcal{P}(\mathbb{S} \times \mathbb{O})$ , where  $\mu \in \mathcal{P}(\mathbb{S})$ . We therefore introduce the following notation: for any  $\mu \in \mathcal{P}(\mathbb{S})$ , we define  $\mathbf{P}^\mu = \mathbf{P}_{\mu \times \delta_{\{0\}}}$ . Similarly,  $\mathbf{E}^\mu$  denotes the expectation with respect to  $\mathbf{P}^\mu$ .

*Remark 2.1.* There is no loss of generality in assuming that  $Y_0 = 0$ . Indeed, consider an arbitrary initial measure  $\mu \in \mathcal{P}(\mathbb{S} \times \mathbb{O})$ . Then by [9, lemma 2.4]

$$\mathbf{E}_\mu(f(X_t) | \mathcal{F}_t^Y) = \mathbf{E}_{\mu(\cdot | Y_0) \times \delta_{\{Y_0\}}} (f(X_t) | \mathcal{F}_t^Y),$$

where  $\mu(\cdot | Y_0)$  is a regular conditional probability of  $X_0$  with respect to  $Y_0$  under  $\mu$ . But note that under any initial measure of the form  $\nu \times \delta_{\{a\}}$ , our assumptions imply that  $\mathcal{F}^X \vee \mathcal{G}^Y$  is independent of  $\mathcal{F}_0^Y$ , so that

$$\mathbf{E}_\mu(f(X_t) | \mathcal{F}_t^Y) = \mathbf{E}_{\mu(\cdot | Y_0) \times \delta_{\{Y_0\}}} (f(X_t) | \mathcal{G}_t^Y) = \mathbf{E}^{\mu(\cdot | Y_0)} (f(X_t) | \mathcal{F}_t^Y)$$

provided that we choose an appropriate version of the latter conditional expectation that is defined  $\mathbf{P}_\mu$ -a.s. Thus it suffices to consider the case  $Y_0 = 0$ .

### 3. SPACES OF OBSERVABLE FUNCTIONS AND NONOBSERVABLE MEASURES

Broadly speaking, the goal of this section is to investigate the following question: what is the relation between the law of  $X_0$  and the law of  $\mathcal{F}^Y$ ? In the next section, we will see that this question has immediate consequences for filter stability.

**Definition 3.1.** For  $\mu, \nu \in \mathcal{P}(\mathbb{S})$ , we write  $\mu \sim \nu$  whenever  $\mathbf{P}^\mu|_{\mathcal{F}^Y} = \mathbf{P}^\nu|_{\mathcal{F}^Y}$ . In particular,  $\sim$  defines an equivalence relation on  $\mathcal{P}(\mathbb{S})$ .

In words, if  $\mu \sim \nu$ , then whenever  $X_0$  has the law  $\mu$  or  $\nu$ , the same law of the observation process is obtained. In particular, no amount of statistics gathered from the observation process will allow us to distinguish between  $X_0 \sim \mu$  and  $X_0 \sim \nu$ . This motivates the following notion of observability, which is reminiscent (at least in spirit) of the notion of observability used in linear systems theory.

**Definition 3.2.** The filtering model is called *observable* if  $\mu \sim \nu$  implies  $\mu = \nu$ .

The following definition is key (we use the notation  $\mu(f) = \int f(x) \mu(dx)$ ).

**Definition 3.3.** Define the *space of nonobservable measures*  $\mathcal{N}$  as

$$\mathcal{N} = \{\alpha\mu_1 - \alpha\mu_2 \in \mathcal{M}(\mathbb{S}) : \alpha \in \mathbb{R}, \mu_1, \mu_2 \in \mathcal{P}(\mathbb{S}), \mu_1 \sim \mu_2\}.$$

Moreover, we define the *space of observable functions*  $\mathcal{O}$  as

$$\mathcal{O} = \{f \in \mathcal{C}_b(\mathbb{S}) : \mu_1(f) = \mu_2(f) \text{ for all } \mu_1 \sim \mu_2\}.$$

We attach to the nonobservable space  $\mathcal{N}$  the following intuitive interpretation: if we perturb the initial measure  $X_0 \sim \mu$  in the direction  $\delta \in \mathcal{N}$  ( $\mu \mapsto \mu + \delta$ , provided  $\mu + \delta$  is again a probability measure), then the law of the observation process does not change. The observable space  $\mathcal{O}$  then consists of those functions  $f$  such that the expectation of  $f(X_0)$  is completely determined by the law of the observation process. Note that the filtering model is observable if and only if every continuous function is observable, i.e.,  $\mathcal{O} = \mathcal{C}_b(\mathbb{S})$ , or, equivalently, if no nontrivial signed measure is nonobservable, i.e.,  $\mathcal{N} = \{0\}$ .

Our goal is to characterize the space  $\mathcal{O}$ . Before we do this, let us recall a simple functional analytic device which will be needed below [30, chapter 4]. Let  $B$  be a Banach space and denote by  $B^*$  its topological dual. Consider two (not necessarily closed) linear subspaces  $M \subset B$  and  $N \subset B^*$ .

**Definition 3.4.** The *annihilator*  $M^\perp \subset B^*$  of  $M$  is defined as

$$M^\perp = \{x^* \in B^* : \langle x^*, x \rangle = 0 \text{ for all } x \in M\}.$$

Similarly, the annihilator  $N^\perp \subset B$  of  $N$  is defined as

$$N^\perp = \{x \in B : \langle x^*, x \rangle = 0 \text{ for all } x^* \in N\}.$$

The proof of the following lemma [30, theorem 4.7] follows from a straightforward application of the Hahn-Banach theorem.

**Lemma 3.5.**  $(M^\perp)^\perp = \overline{M}$ , where  $\overline{M}$  is the (norm-)closure of  $M$  in  $B$ .

Recall that  $\mathcal{M}(\mathbb{S})$  is the topological dual of  $\mathcal{C}_b(\mathbb{S})$  by the Riesz-Markov theorem. It is thus easily verified from the definitions that  $\mathcal{O} = \mathcal{N}^\perp$ . What we will show is that there is a dense subset  $\mathcal{O}^0 \subset \mathcal{O}$  such that every  $f \in \mathcal{O}^0$  can be written as  $f(x) = \mathbf{E}_{(x,y)}(\xi)$  for some bounded  $\mathcal{G}^Y$ -measurable random variable  $\xi$ .



**Proposition 3.6.** *Let  $\mathcal{O}^0$  be the linear span of functions of the form*

$$\mathbf{E}_{(x,y)}(f_1(Y_{t_1} - Y_0)f_2(Y_{t_2} - Y_0) \cdots f_n(Y_{t_n} - Y_0)),$$

*for all  $n < \infty$ ,  $t_i \in D$  and bounded continuous functions  $f_i$  on  $\mathbb{O}$ , where  $D$  is a dense subset of  $[0, \infty[$ . Then  $\mathcal{O}^0$  is dense in  $\mathcal{O}$ . In particular, for any observable function  $f \in \mathcal{O}$ , there is a sequence of functions  $f_n \in \mathcal{O}^0$  such that  $\|f - f_n\| \rightarrow 0$ .*

*Proof.* By our assumptions, any  $f \in \mathcal{O}^0$  only depends on  $\mathbb{S}$ , and we find

$$\begin{aligned} \mathbf{E}_{(x,y)}(f_1(Y_{t_1} - Y_0)f_2(Y_{t_2} - Y_0) \cdots f_n(Y_{t_n} - Y_0)) &= \\ \mathbf{E}_{(x,0)}(f_1(Y_{t_1} - Y_0)f_2(Y_{t_2} - Y_0) \cdots f_n(Y_{t_n} - Y_0)) &= \\ \mathbf{E}_{(x,0)}(f_1(Y_{t_1})f_2(Y_{t_2}) \cdots f_n(Y_{t_n})). \end{aligned}$$

We claim that  $\mathcal{O}^0 \subset \mathcal{C}_b(\mathbb{S})$ . As  $\mathbb{S}$  is compact, it suffices to show that any  $f \in \mathcal{O}^0$  is continuous. But if  $x_n \rightarrow x$  in  $\mathbb{S}$ , then by [19, theorem 17.25] the measures  $\mathbf{P}_{(x_n,0)}$  converge weakly to  $\mathbf{P}_{(x,0)}$ , and this in turn implies weak convergence of the finite dimensional distributions on some dense subset of times  $D$  [14, theorem 3.7.8]. The continuity of  $f \in \mathcal{O}^0$  follows directly from the previous expression.

To show that  $\mathcal{O}^0$  is dense in  $\mathcal{O}$ , it suffices to show that  $(\mathcal{O}^0)^\perp = \mathcal{N}$  by lemma 3.5. Note that by [5, theorem 16.6] the finite dimensional distributions in a dense set of times form a separating class for probability measures on  $D([0, \infty[; \mathbb{O})$ . Hence a standard monotone class argument shows that  $\mu_1 \sim \mu_2$  if and only if  $\mathbf{E}^{\mu_1}(f_1(Y_{t_1})f_2(Y_{t_2}) \cdots f_n(Y_{t_n})) = \mathbf{E}^{\mu_2}(f_1(Y_{t_1})f_2(Y_{t_2}) \cdots f_n(Y_{t_n}))$  for all finite sets of times  $t_i \in D$  and bounded continuous  $f_i$ . But using the previous equation display, this is clearly the case if and only if  $\mu_1(f) = \mu_2(f)$  for all  $f \in \mathcal{O}^0$ . Hence we find immediately that  $\mathcal{N} \subset (\mathcal{O}^0)^\perp$ . On the other hand, choose any  $\mu \in (\mathcal{O}^0)^\perp$  (with  $\mu \neq 0$ ), and define  $\mu_1 = \mu^+/\alpha$ ,  $\mu_2 = \mu^-/\alpha$  with  $\alpha = \mu^+(\mathbb{S})$  (here  $\mu = \mu^+ - \mu^-$  is the Hahn decomposition of  $\mu$ ). Note that  $\mu_1$  and  $\mu_2$  are both probability measures (due to the fact that  $1 \in \mathcal{O}^0$  implies  $\mu(\mathbb{S}) = 0$ ), and  $\mu = \alpha\mu_1 - \alpha\mu_2$ . But  $\mu_1(f) = \mu_2(f)$  for all  $f \in \mathcal{O}^0$  implies  $\mu_1 \sim \mu_2$ , so evidently  $\mu \in \mathcal{N}$ . Hence we have established the converse inclusion  $\mathcal{N} \supset (\mathcal{O}^0)^\perp$ , and the proof is complete.  $\square$

*Remark 3.7.* One might hope that *any* observable function  $f \in \mathcal{O}$  can be written as  $f(x) = \mathbf{E}_{(x,y)}(\xi)$  for some bounded  $\mathcal{G}^Y$ -measurable  $\xi$ . This seemingly plausible conjecture need not hold true, however, as the following simplified example illustrates. Let  $X$  be a  $[0, 1]$ -valued random variable with law  $\mu$ , and let  $Y = X + \xi$  where  $\xi$  is Gaussian with zero mean and unit variance. Denote by  $\mathbf{P}_\mu$  the joint law of  $X$  and  $Y$ . Then the same argument used in the previous proof shows that any continuous function  $f : [0, 1] \rightarrow \mathbb{R}$  can be written as the uniform limit of functions of the form  $f_n(x) = \mathbf{E}_{\delta_{\{x\}}}(g_n(Y))$ . However, any  $f_n$  will necessarily be a smooth function (being the convolution of the bounded function  $g_n$  with the Gaussian density), so that evidently not all  $f$  can be expressed in this form. Thus in general, an approximation result is the best one could hope for.

#### 4. FILTER STABILITY AND OBSERVABILITY

We now connect the notions of observability introduced in the previous section to the stability of the nonlinear filter. Recall that we are interested in determining, given a pair of initial measures  $\mu, \nu$ , whether  $\mathbf{E}^\mu(f(X_t)|\mathcal{F}_t^Y)$  and  $\mathbf{E}^\nu(f(X_t)|\mathcal{F}_t^Y)$  are close to each other for large times  $t$ . We will see that this is always the case when the function  $f$  is observable, i.e., when  $f \in \mathcal{O}$ , provided that  $\mu \ll \nu$ .

The following lemma, which is inspired by a result of Chigansky and Liptser [8, theorem 2.1], contains the essence of the convergence argument.

**Lemma 4.1.** *Let  $\mu, \nu \in \mathcal{P}(\mathbb{S})$  satisfy  $\mu \ll \nu$ . Moreover, let  $(\xi_t)_{t \in [0, \infty[}$  be  $\mathcal{F}^Y$ -measurable random variables with  $|\xi_t| \leq K < \infty$  for all  $t$ , such that the sample paths  $t \mapsto \xi_t$  are càdlàg. Then we have*

$$|\mathbf{E}^\mu(\xi_t | \mathcal{F}_t^Y) - \mathbf{E}^\nu(\xi_t | \mathcal{F}_t^Y)| \xrightarrow{t \rightarrow \infty} 0 \quad \mathbf{P}^\mu\text{-a.s.}$$

*Remark 4.2.* Recall that whenever conditional expectations are encountered, the corresponding càdlàg versions are implied. Throughout the following proofs, we will use the usual properties of conditional expectations to obtain equalities and inequalities that, for every time  $t$ , hold for all  $\omega \in \Omega \setminus N_t$  where  $N_t$  is a  $\mathbf{P}^\mu$ -null set. Because all the processes are càdlàg, however, the null set can be chosen independent of time  $t$ , so that these equalities and inequalities hold for all  $t$  simultaneously with unit probability. We will use this fact below without further comment.

*Proof.* We begin by noting that [9, lemma 2.1]

$$\frac{d\mathbf{P}^\mu}{d\mathbf{P}^\nu} = \frac{d\mu}{d\nu}(X_0).$$

By the Bayes formula, we obtain  $\mathbf{P}^\mu$ -a.s.

$$\mathbf{E}^\nu \left( \frac{d\mu}{d\nu}(X_0) \middle| \mathcal{F}_t^Y \right) \mathbf{E}^\mu(\xi_t | \mathcal{F}_t^Y) = \mathbf{E}^\nu \left( \frac{d\mu}{d\nu}(X_0) \xi_t \middle| \mathcal{F}_t^Y \right).$$

Introduce the notation

$$\varrho_t = \mathbf{E}^\nu \left( \frac{d\mu}{d\nu}(X_0) \middle| \mathcal{F}_t^Y \right), \quad \varrho_\infty = \mathbf{E}^\nu \left( \frac{d\mu}{d\nu}(X_0) \middle| \mathcal{F}^Y \right).$$

Then we find, using the fact that  $\xi_t$  is  $\mathcal{F}^Y$ -measurable,

$$\varrho_t |\mathbf{E}^\mu(\xi_t | \mathcal{F}_t^Y) - \mathbf{E}^\nu(\xi_t | \mathcal{F}_t^Y)| = |\mathbf{E}^\nu((\varrho_\infty - \varrho_t) \xi_t | \mathcal{F}_t^Y)| \leq K \mathbf{E}^\nu(|\varrho_\infty - \varrho_t| | \mathcal{F}_t^Y).$$

That this expression converges to zero  $\mathbf{P}^\mu$ -a.s. is established in lemma 4.3 below. But as  $\varrho_t \rightarrow \varrho_\infty$   $\mathbf{P}^\nu$ -a.s. by Lévy's upward theorem, we conclude the convergence  $|\mathbf{E}^\mu(\xi_t | \mathcal{F}_t^Y) - \mathbf{E}^\nu(\xi_t | \mathcal{F}_t^Y)| \rightarrow 0$  as  $t \rightarrow \infty$  on  $\{\omega \in \Omega : \varrho_\infty(\omega) > 0\} \in \mathcal{F}^Y$  (modulo a  $\mathbf{P}^\mu$ -null set), and the latter set has  $\mathbf{P}^\mu$ -measure one.  $\square$

The proof of the previous lemma is not yet complete, as we still need to show that  $\mathbf{E}^\nu(|\varrho_\infty - \varrho_t| | \mathcal{F}_t^Y) \rightarrow 0$ . If we were interested in  $L^1$  convergence rather than a.s. convergence, the result is trivially established. Proving a.s. convergence would appear to be a matter of applying Hunt's lemma [12, chapter V, theorem 45], whose proof is easily adapted to the continuous time setting. Unfortunately, this would require  $\varrho_t$  to be dominated by an integrable random variable, which may not be the case (to guarantee that this is the case we could impose, e.g., a finite relative entropy condition  $D(\mu | \nu) < \infty$ , see [12, chapter V, sec. 25(c)]). Instead, we proceed by adapting Rao's proof of Hunt's lemma [28, lemma 2] to our setting.

**Lemma 4.3.**  $\mathbf{E}^\nu(|\varrho_\infty - \varrho_t| | \mathcal{F}_t^Y) \xrightarrow{t \rightarrow \infty} 0$   $\mathbf{P}^\nu$ -a.s.

*Proof.* Denote  $|\varrho_\infty - \varrho_t| = u_t$  and  $\mathbf{E}^\nu(|\varrho_\infty - \varrho_t| | \mathcal{F}_t^Y) = v_t$ , and fix a constant  $\varepsilon > 0$ . Define the following stopping times:

$$\tau_1 = \inf\{t > 0 : v_t > \varepsilon\}, \quad \sigma_1 = \inf\{t > \tau_1 : v_t < \varepsilon/2\},$$



and for any  $n \geq 2$

$$\tau_n = \inf\{t > \sigma_{n-1} : v_t > \varepsilon\}, \quad \sigma_n = \inf\{t > \tau_n : v_t < \varepsilon/2\}.$$

By right-continuity of the sample paths,  $v_{\tau_n} \geq \varepsilon$  on  $\{\tau_n < \infty\}$ . But then

$$\varepsilon \mathbf{P}^\nu(\tau_n < \infty) = \varepsilon \mathbf{P}^\nu(v_{\tau_n} I_{\tau_n < \infty} \geq \varepsilon) \leq \mathbf{E}^\nu(v_{\tau_n} I_{\tau_n < \infty}),$$

where we have used Chebyshev's inequality. But as  $v_t$  is the optional projection of  $u_t$  [12, chapter VI, theorems 43 and 47], we can write  $v_{\tau_n} I_{\tau_n < \infty} = \mathbf{E}^\nu(u_{\tau_n} I_{\tau_n < \infty} | \mathcal{F}_{\tau_n}^Y)$   $\mathbf{P}^\nu$ -a.s. Hence, in particular,  $\varepsilon \mathbf{P}^\nu(\tau_n < \infty) \leq \mathbf{E}^\nu(u_{\tau_n} I_{\tau_n < \infty})$ .

We now claim that  $\tau_n \rightarrow \infty$  as  $n \rightarrow \infty$  a.s. To see this, note that  $\tau_n$  is nondecreasing, so it must converge either to infinity or to a finite value. But if it converges to a finite value, then that sample path of  $Y_t$  must have a discontinuity of the second kind and hence cannot be càdlàg. Thus we can conclude that  $\tau_n \rightarrow \infty$  a.s., and hence  $u_{\tau_n} \rightarrow 0$  a.s. by Lévy's upward theorem (as  $\varrho_t \rightarrow \varrho_\infty$  a.s.). We would like to show that  $u_{\tau_n} \rightarrow 0$  in  $L^1$ , so that we can conclude that  $\mathbf{P}^\nu(\tau_n < \infty) \rightarrow 0$  as  $n \rightarrow \infty$ . To this end, note that  $u_{\tau_n} \rightarrow 0$  in  $L^1$  is equivalent to  $\varrho_{\tau_n} \rightarrow \varrho_\infty$  in  $L^1$ . But applying again the optional projection property, we find that  $\varrho_{\tau_n} = \mathbf{E}^\nu(\varrho_\infty | \mathcal{F}_{\tau_n}^Y)$   $\mathbf{P}^\nu$ -a.s. Hence the desired convergence follows from Lévy's upward theorem.

We have established that  $\mathbf{P}^\nu(\tau_n < \infty) \rightarrow 0$  as  $n \rightarrow \infty$ . It follows directly that  $\mathbf{P}^\nu(\tau_n < \infty \text{ for all } n) \leq \inf_n \mathbf{P}^\nu(\tau_n < \infty) = 0$ , so with unit probability either  $\limsup_{n \rightarrow \infty} v_t \leq \varepsilon$ , or  $\liminf_{n \rightarrow \infty} v_t \geq \varepsilon/2$ . But note that  $\|v_t\|_1 \leq \|u_t\|_1 \rightarrow 0$  as  $t \rightarrow \infty$ , so  $v_t \rightarrow 0$  in  $L^1$ . Hence  $\liminf_{n \rightarrow \infty} v_t \geq \varepsilon/2$  can only happen on a null set, and we conclude that  $\limsup_{n \rightarrow \infty} v_t \leq \varepsilon$  a.s. As this holds for any  $\varepsilon > 0$ , the desired convergence is established.  $\square$

We are finally in a position to prove the main result.

**Theorem 4.4.** *Let  $\mu \ll \nu$  and  $f \in \mathcal{O}$ . Then*

$$|\mathbf{E}^\mu(f(X_t) | \mathcal{F}_t^Y) - \mathbf{E}^\nu(f(X_t) | \mathcal{F}_t^Y)| \rightarrow 0 \quad \mathbf{P}^\mu\text{-a.s.}$$

*Proof.* First, note that it suffices to prove the theorem for  $f \in \mathcal{O}^0$ . After all, suppose we have established the result for  $\mathcal{O}^0$ . By proposition 3.6, there is for  $f \in \mathcal{O}$  a sequence  $f_n \in \mathcal{O}^0$  such that  $\|f - f_n\| \rightarrow 0$  as  $n \rightarrow \infty$ . Then

$$\begin{aligned} & \limsup_{t \rightarrow \infty} |\mathbf{E}^\mu(f(X_t) | \mathcal{F}_t^Y) - \mathbf{E}^\nu(f(X_t) | \mathcal{F}_t^Y)| \\ & \leq \limsup_{t \rightarrow \infty} |\mathbf{E}^\mu(f(X_t) - f_n(X_t) | \mathcal{F}_t^Y)| + \limsup_{t \rightarrow \infty} |\mathbf{E}^\nu(f_n(X_t) - f(X_t) | \mathcal{F}_t^Y)| \\ & \quad + \limsup_{t \rightarrow \infty} |\mathbf{E}^\mu(f_n(X_t) | \mathcal{F}_t^Y) - \mathbf{E}^\nu(f_n(X_t) | \mathcal{F}_t^Y)| \\ & \leq 2\|f - f_n\| + \limsup_{t \rightarrow \infty} |\mathbf{E}^\mu(f_n(X_t) | \mathcal{F}_t^Y) - \mathbf{E}^\nu(f_n(X_t) | \mathcal{F}_t^Y)| \\ & = 2\|f - f_n\| \quad \mathbf{P}^\mu\text{-a.s.} \end{aligned}$$

But then the result follows for  $f \in \mathcal{O}$  by letting  $n \rightarrow \infty$ .

We may thus assume that  $f \in \mathcal{O}^0$ , and by the linearity of the conditional expectation we may assume without loss of generality that  $f$  is of the form

$$f(x) = \mathbf{E}_{(x,y)}(\xi), \quad \xi = f_1(Y_{t_1} - Y_0) f_2(Y_{t_2} - Y_0) \cdots f_n(Y_{t_n} - Y_0),$$

for some  $n < \infty$ ,  $t_i \in D$  and bounded continuous functions  $f_i$ . By the Markov property, we find that  $f(X_t) = \mathbf{E}^\nu(\xi \circ \theta_t | \mathcal{F}_t)$   $\mathbf{P}^\nu$ -a.s. and that  $f(X_t) = \mathbf{E}^\mu(\xi \circ \theta_t | \mathcal{F}_t)$

$\mathbf{P}^\mu$ -a.s., so we obtain

$$|\mathbf{E}^\mu(f(X_t)|\mathcal{F}_t^Y) - \mathbf{E}^\nu(f(X_t)|\mathcal{F}_t^Y)| = |\mathbf{E}^\mu(\xi \circ \theta_t|\mathcal{F}_t^Y) - \mathbf{E}^\nu(\xi \circ \theta_t|\mathcal{F}_t^Y)|.$$

But as the  $f_i$  are continuous,  $\xi_t = \xi \circ \theta_t$  has càdlàg sample paths, and clearly  $\xi_t$  is  $\mathcal{F}^Y$ -measurable for every  $t$ . It remains to apply lemma 4.1.  $\square$

An immediate consequence is that observability implies stability.

**Definition 4.5.** A filtering model is *stable* if whenever  $\mu \ll \nu$ ,

$$|\mathbf{E}^\mu(f(X_t)|\mathcal{F}_t^Y) - \mathbf{E}^\nu(f(X_t)|\mathcal{F}_t^Y)| \rightarrow 0 \quad \mathbf{P}^\mu\text{-a.s.} \quad \text{for all } f \in \mathcal{C}_b(\mathbb{S}).$$

**Corollary 4.6.** *If the filtering model is observable, then it is stable.*

*Proof.* This is immediate from the definition of observability.  $\square$

*Remark 4.7.* A word should be said at this point about the assumptions that the signal process is a Markov process and that the observation process has conditionally independent increments. There is nothing essential in the convergence proofs that depends on these properties, and indeed these can safely be dropped (in fact, one may then choose the observation state space  $\mathbb{O}$  to be any locally compact Polish space). In this case, however, we could not guarantee that the space of observable functions will contain only functions on  $\mathbb{S}$ ; instead, we would obtain  $\mathcal{O} \subset \mathcal{C}_0(\mathbb{S} \times \mathbb{O})$  and  $\mathcal{N} \subset \mathcal{M}(\mathbb{S} \times \mathbb{O})$ , and we would have to consider convergence of conditional expectations of the form  $\mathbf{E}(f(X_t, Y_t)|\mathcal{F}_t^Y)$ . In other words, in this case the initial measure on the observation process can play a nontrivial role, which is not surprising. The setting in which we have chosen to work—where the signal dynamics does not depend on the observations and the observation noise is memoryless—is the natural setting where the initial measure on the observations decouples from the problem. This allows us to concentrate on filtered estimates of the signal process, which are the quantities which are of interest in the majority of applications.

*Remark 4.8.* Our notion of stability requires that  $\mu \ll \nu$ . This is unavoidable if we wish to define the filtered estimates as conditional expectations: as  $\mathbf{E}^\mu(f(X_t)|\mathcal{F}_t^Y)$  is only defined up to  $\mathbf{P}^\mu$ -a.s. equivalence, the comparison of  $\mathbf{E}^\mu(f(X_t)|\mathcal{F}_t^Y)$  and  $\mathbf{E}^\nu(f(X_t)|\mathcal{F}_t^Y)$  for  $\mu \not\ll \nu$  need not make sense under any measure. In many cases, however, there is a natural version of the conditional expectations which may be defined simultaneously with respect to all  $\mathbf{P}^\nu$ . In this case, one may ask whether the filter is *strong stable*, i.e., whether stability holds even for  $\mu \not\ll \nu$ . This typically requires a *controllability* assumption in addition to observability (section 7.3). For the time being we are chiefly interested in observability, but we will return to the strong stability problem in section 7 in the setting of white noise type observations.

## 5. WHITE NOISE TYPE AND COUNTING OBSERVATIONS

The purpose of this section is to investigate how two specific observation models that are extremely common in practice—white noise and counting observations—fit into the general results developed in the previous subsections.

**5.1. White noise type observations.** We consider the following setting:  $X_t$  is a Feller-Markov process, and  $Y_t$  can be written in the form

$$Y_t = Y_0 + \int_0^t h(X_s) ds + KB_t,$$

where  $K$  is a non-random  $p \times q$  matrix,  $h : \mathbb{S} \rightarrow \mathbb{O}$  is a continuous function, and  $KB_t$  is a  $p$ -dimensional Wiener process, with covariance matrix  $KK^*$ , which is independent of  $X_t$  and  $Y_0$  (for any  $\mathbf{P}_\nu$ ). Note that  $K$  may be degenerate, in which case  $v^*KB_t$  could be identically zero for certain  $v \in \mathbb{R}^p$ .

**Lemma 5.1.** *The white noise type observation model satisfies the conditionally independent increments property.*

*Proof.* Let  $\xi$  be any bounded,  $\mathcal{F}^X \vee \mathcal{G}^Y$ -measurable random variable. Then  $\xi$  is  $\mathcal{F}^X \vee \sigma\{KB_t : t > 0\}$ -measurable. We claim that for any  $\mathcal{F}^X \vee \sigma\{KB_t : t > 0\}$ -measurable random variable,  $\mathbf{E}_{(x,y)}(\xi)$  is independent of  $y$ . To establish this, it suffices to prove the claim for functions of the form  $\xi_1\xi_2$  where  $\xi_1$  is  $\mathcal{F}^X$ -measurable and  $\xi_2$  is  $\sigma\{KB_t : t > 0\}$ -measurable; the statement then follows by the monotone class theorem. But  $\mathbf{E}_{(x,y)}(\xi_1\xi_2) = \mathbf{E}_{(x,y)}(\xi_1)\mathbf{E}_{(x,y)}(\xi_2)$  by independence, while  $\mathbf{E}_{(x,y)}(\xi_1)$  only depends on  $x$  (as  $X_t$  is a Markov process) and  $\mathbf{E}_{(x,y)}(\xi_2)$  depends on neither  $x$  or  $y$  (as  $B_t$  is a Wiener process for every  $\mathbf{P}_\nu$ ).  $\square$

As the observations only depend on the signal through the observation function  $h$ , a natural question is whether the observable and nonobservable spaces depend on the noise covariance  $KK^*$ . As one might expect, this is not the case; for the purpose of observability, we may simply take  $K = 0$ . This is very convenient in computations, and shows that observability is a structural property which does not depend on the signal-to-noise ratio of the observations.

**Proposition 5.2.** *For the white noise type observation model,*

$$\mathcal{N} = \{\alpha\mu_1 - \alpha\mu_2 \in \mathcal{M}(\mathbb{S}) : \alpha \in \mathbb{R}, \mu_1, \mu_2 \in \mathcal{P}(\mathbb{S}), \mathbf{P}^{\mu_1}|_{\mathcal{G}^h} = \mathbf{P}^{\mu_2}|_{\mathcal{G}^h}\},$$

where  $\mathcal{G}^h = \sigma\{h(X_t) : t \geq 0\}$ .

To prove this statement, we will need the following simple lemma.

**Lemma 5.3.** *Let  $(Z_1, \dots, Z_n)$  and  $(Z'_1, \dots, Z'_n)$  be arbitrary random variables, and let  $(\xi_1, \dots, \xi_n)$  be Gaussian random variables independent of all  $Z_i, Z'_i$ . Then*

$$(Z_1, \dots, Z_n) \stackrel{\text{law}}{=} (Z'_1, \dots, Z'_n) \quad \text{iff} \quad (Z_1 + \xi_1, \dots, Z_n + \xi_n) \stackrel{\text{law}}{=} (Z'_1 + \xi_1, \dots, Z'_n + \xi_n).$$

*Proof.* Recall that a probability measure on  $\mathbb{R}^n$  is uniquely determined by its characteristic function. Denote by  $\chi_Z, \chi_{Z'}, \chi_{Z+\xi}, \chi_{Z'+\xi}$ , and  $\chi_\xi$  the characteristic functions of  $(Z_1, \dots, Z_n), (Z'_1, \dots, Z'_n), (Z_1 + \xi_1, \dots, Z_n + \xi_n), (Z'_1 + \xi_1, \dots, Z'_n + \xi_n)$ , and  $(\xi_1, \dots, \xi_n)$ , respectively. Then, by independence,  $\chi_{Z+\xi} = \chi_Z\chi_\xi$  and  $\chi_{Z'+\xi} = \chi_{Z'}\chi_\xi$ . But as  $\xi$  is a Gaussian random vector,  $\chi_\xi$  is invertible, so evidently  $\chi_{Z+\xi} = \chi_{Z'+\xi}$  iff  $\chi_Z = \chi_{Z'}$ . This establishes the claim.  $\square$

*Proof of proposition 5.2.* Recall that  $\mu \in \mathcal{N}$  iff there exist probability measures  $\mu_1, \mu_2 \in \mathcal{P}(\mathbb{S})$  and  $\alpha > 0$  such that  $\mu = \alpha\mu_1 - \alpha\mu_2$  and  $\mathbf{P}^{\mu_1}|_{\mathcal{F}^Y} = \mathbf{P}^{\mu_2}|_{\mathcal{F}^Y}$ . By [5, theorem 16.6] the finite dimensional distributions form a separating class for probability measures on  $D([0, \infty[; \mathbb{O})$ , so  $\mu \in \mathcal{N}$  iff  $\mathbf{P}^{\mu_1}|_{\mathcal{G}} = \mathbf{P}^{\mu_2}|_{\mathcal{G}}$  for all  $\mathcal{G} = \sigma\{Y_{t_1}, \dots, Y_{t_n}\}$  with  $n < \infty$  and  $t_1, \dots, t_n \in [0, \infty[$ . But by lemma 5.3, this is the case iff  $\mathbf{P}^{\mu_1}|_{\mathcal{H}} = \mathbf{P}^{\mu_2}|_{\mathcal{H}}$  for all  $\mathcal{H} = \sigma\{\int_0^{t_1} h(X_s) ds, \dots, \int_0^{t_n} h(X_s) ds\}$  with  $n < \infty$  and  $t_1, \dots, t_n \in [0, \infty[$ . As  $\int_0^t h(X_s) ds$  is continuous (and in particular càdlàg), so that the finite dimensional distributions form a separating class also for this process, and as  $\sigma\{\int_0^t h(X_s) ds : t > 0\} = \mathcal{G}^h$ , the result follows.  $\square$

**5.2. Counting observations.** We now turn to the case of counting observations, for which almost identical results hold. In this setting,  $X_t$  is again a Feller-Markov process, and  $Y_t = Y_0 + N_t$  where  $N_t$  is a Cox process [19, proposition 10.5] with intensity  $\lambda_t = h(X_t)$ , conditionally independent of  $Y_0$  given  $\mathcal{F}^X$ , for every  $\mathbf{P}_\mu$ . By definition, this implies that for any  $\mu$ , under a regular conditional probability  $\mathbf{P}_\mu(\cdot | \mathcal{F}^X)$  (which exists as our spaces are Polish),  $N_t^i$  are independent Poisson processes with intensities  $\lambda_t^i$  and the process  $N_t$  is independent of  $Y_0$ . Here the observation function  $h : \mathbb{S} \rightarrow \mathbb{O}$  is a continuous nonnegative function.

**Lemma 5.4.** *The counting observation model satisfies the conditionally independent increments property.*

*Proof.* Let  $\xi$  be any bounded,  $\mathcal{F}^X \vee \mathcal{G}^Y$ -measurable random variable. Then by our assumptions, under a regular conditional probability  $\mathbf{P}_{(x,y)}(\cdot | \mathcal{F}^X)$ , the law of  $\xi$  only depends on the sample paths of  $X_t$  and is thus independent of  $y$ . In particular, this means that  $\mathbf{E}_{(x,y)}(\xi | \mathcal{F}^X)$  does not depend on  $y$ . But then  $\mathbf{E}_{(x,y)}(\xi) = \mathbf{E}_{(x,y)}(\mathbf{E}_{(x,y)}(\xi | \mathcal{F}^X))$  can not depend on  $y$ , as  $X_t$  is a Markov process in its own right and as  $\mathbf{E}_{(x,y)}(\xi | \mathcal{F}^X)$  is an  $\mathcal{F}^X$ -measurable random variable.  $\square$

An analog of proposition 5.2 also holds.

**Proposition 5.5.** *The conclusion of proposition 5.2 holds identically for the counting observation model.*

*Proof.* This follows directly from [19, lemma 10.8].  $\square$

**5.3. A simple sufficient condition.** Let us mention a useful consequence of these results, which leads to a particularly simple sufficient condition for observability.

**Lemma 5.6.** *For the white noise type and counting observations models, it is always the case that  $f \circ h \in \mathcal{O}$  for any measurable function  $f : \mathbb{O} \rightarrow \mathbb{R}$  such that  $f \circ h \in \mathcal{C}_b(\mathbb{S})$ . In particular, if  $h$  is one-to-one then we may conclude that  $\mathcal{O} = \mathcal{C}_b(\mathbb{S})$  (i.e., the signal-observation model is observable).*

*Proof.* Let  $f \circ h \in \mathcal{C}_b(\mathbb{S})$ , and choose any  $\mu \in \mathcal{N}$ . Then  $\mu = \alpha\mu_1 - \alpha\mu_2$ , where  $\mathbf{P}^{\mu_1}|_{\mathcal{G}^h} = \mathbf{P}^{\mu_2}|_{\mathcal{G}^h}$ . Thus in particular  $\mathbf{P}^{\mu_1}|_{\sigma\{h(X_0)\}} = \mathbf{P}^{\mu_2}|_{\sigma\{h(X_0)\}}$ . But  $f(h(X_0))$  is  $\sigma\{h(X_0)\}$ -measurable, so that evidently

$$\int f(h(x)) \mu_1(dx) = \int f(h(x)) \mu_2(dx).$$

As this holds for any  $\mu \in \mathcal{N}$ , we find that  $f \circ h \in \mathcal{O}$ .  $\square$

In other words, “nice” functions of the observation function are always observable, regardless of any further properties of the model.

*Remark 5.7.* For the special case where  $f$  is chosen to be the identity, the stability of the observation function (in a slightly different sense) was found in [9, theorem 3.1] under much weaker conditions. However, the latter result cannot be used to conclude the stability of the filter, even in the case when  $h$  is one-to-one.

## 6. FINITE STATE SIGNALS

The simplest nonlinear filtering model is one where the signal state space consists of a finite number of points. Such models are of particular theoretical and practical interest as the filtered estimates can be finite dimensionally computed. On the other hand, this model shares many of the features of more general models and thus serves as a convenient prototype. In this section, we will use the results of the previous section to obtain an essentially complete characterization of the stability of such filters in the case of nondegenerate white noise type observations.

Throughout this section,  $X_t$  is a Markov process on the finite state space  $\mathbb{S} = \{a_1, \dots, a_d\}$  with transition intensities matrix  $\Lambda = (\lambda_{ij})_{1 \leq i, j \leq d}$ . The observations process  $Y_t$  is taken to be one-dimensional and of the form

$$Y_t = Y_0 + \int_0^t h(X_s) ds + \kappa W_t, \quad Y_0 = 0,$$

where  $h : \mathbb{S} \rightarrow \mathbb{R}$ . The restriction to one dimension is for notational convenience only; all the results extend directly to observations in  $\mathbb{R}^p$ .

*Remark 6.1.*  $Y_0$  is included as a reminder that this is a Markov observation model. As usual, we will enforce  $Y_0 = 0$  by working with the measures  $\mathbf{P}^\mu$ .

In subsection 6.1, we elaborate on the structure of the spaces  $\mathcal{N}$  and  $\mathcal{O}$  in this setting (the results of this section hold identically for the case of counting observations). Subsection 6.2 (see also section 7.2) is devoted to the complete characterization of the stability of the filter. Here we make essential use of the white noise type observations, and it is moreover crucial that the observations are assumed to be *nondegenerate*  $\kappa > 0$ . The reason for this is that we will invoke results that hold only in this setting; see remark 6.13 below for further details.

Finally, a word on notation. We use the following notation for the filter:  $\pi_t^\mu(f) = \mathbf{E}^\mu(f(X_t) | \mathcal{F}_t^Y)$ . When  $\pi_t^\mu$  is used as a vector, this is implied in the sense that  $(\pi_t^\mu)^i = \pi_t^\mu(I_{\{a_i\}})$ . We will interchangeably treat functions on  $\mathbb{S}$  as vectors in  $\mathbb{R}^d$  in the obvious way ( $v^i = v(a_i)$ ), whenever this is convenient. The transpose of a vector or matrix is denoted as  $v^*$  or  $M^*$ .

**6.1. Observability.** For the particular case of a finite state signal and  $\kappa = 0$ , the notion of observability has been investigated in the context of identifiability and lumpability of hidden Markov models [17, 23, 15], though chiefly in discrete time. In this subsection, we briefly develop the necessary results in our setting.

Let  $\mu, \nu \in \mathcal{P}(\mathbb{S})$ . To determine the nonobservable space  $\mathcal{N}$ , by proposition 5.2, we need to find all  $\mu, \nu$  with  $\mathbf{P}^\mu|_{\mathcal{G}^h} = \mathbf{P}^\nu|_{\mathcal{G}^h}$ . But as  $h(X_t)$  is a càdlàg process, it suffices to verify that the finite dimensional distributions of  $h(X_t)$  are the same under  $\mathbf{P}^\mu$  and  $\mathbf{P}^\nu$  [5, theorem 16.6]. We thus begin by computing these distributions.

**Lemma 6.2.** *Let  $\mathbb{H} = h(\mathbb{S}) = \{b_1, \dots, b_r\}$ ,  $r \leq d$  be the set of possible observation values. Define the  $d \times d$  projection matrices  $H_{b_k}$  such that  $(H_{b_k})_{i,j} = 1$  whenever  $i = j$  and  $h(a_i) = b_k$ , and zero otherwise. Then under  $\mathbf{P}^\mu$ , the finite dimensional distributions of  $h(X_t)$  have the form*

$$\mathbf{P}^\mu(h(X_0) = n_0, h(X_{t_1}) = n_1, \dots, h(X_{t_k}) = n_k) = \mu^* H_{n_0} e^{\Lambda t_1} H_{n_1} e^{\Lambda(t_2 - t_1)} H_{n_2} \dots e^{\Lambda(t_k - t_{k-1})} H_{n_k} \mathbf{1},$$

where  $n_i \in \mathbb{H}$  and  $\mathbf{1} \in \mathbb{R}^d$  is the vector of ones.

*Proof.* The result follows from

$$\mathbf{P}^\mu(X_0 = m_0, X_{t_1} = m_1, \dots, X_{t_k} = m_k) = \mu_{m_0}[e^{\Lambda t_1}]_{m_0 m_1} [e^{\Lambda(t_2 - t_1)}]_{m_1 m_2} \cdots [e^{\Lambda(t_k - t_{k-1})}]_{m_{k-1} m_k}.$$

by summing  $m_j$  over the set  $\{a_i \in \mathbb{S} : h(a_i) = n_j\}$ .  $\square$

We immediately conclude the following.

**Corollary 6.3.** *The observable and nonobservable spaces satisfy*

$$\mathcal{O} = \text{span} \{ H_{n_0} e^{\Lambda \delta_1} H_{n_1} e^{\Lambda \delta_2} H_{n_2} \cdots e^{\Lambda \delta_k} H_{n_k} \mathbf{1} : k \geq 0, \delta_i > 0, n_i \in \mathbb{H} \},$$

$$\mathcal{N} = \mathcal{O}^\perp = \{v \in \mathbb{R}^d : v^* x = 0 \text{ for all } x \in \mathcal{O}\}.$$

*The model is observable if and only if  $\dim \mathcal{O} = d$ .*

The following simplification is useful in computations.

**Lemma 6.4.** *The observable space can be characterized as follows:*

$$\mathcal{O} = \text{span} \{ H_{n_0} \Lambda H_{n_1} \Lambda \cdots \Lambda H_{n_k} \mathbf{1} : k \geq 0, n_i \in \mathbb{H} \}.$$

*Proof.* Note that any vector of the form  $(p_i \leq d - 1)$

$$H_{n_0} \Lambda^{p_1} H_{n_1} \Lambda^{p_2} H_{n_2} \cdots \Lambda^{p_k} H_{n_k} \mathbf{1}$$

can be obtained from a vector of the form

$$H_{n_0} e^{\Lambda \delta_1} H_{n_1} e^{\Lambda \delta_2} H_{n_2} \cdots e^{\Lambda \delta_k} H_{n_k} \mathbf{1}$$

by taking derivatives with respect to  $\delta_i$ , and in particular the former is the limit of elements of  $\mathcal{O}$ . But  $\mathcal{O}$  is closed as it is a finite dimensional linear space, so the span of the former is contained in  $\mathcal{O}$ . To prove the converse inclusion, it suffices to expand the matrix exponential in a power series and apply the Cayley-Hamilton theorem. Finally, note that  $H_{b_i}$  sum to the identity matrix, so we can reduce to the case where  $p_i = 1$  for all  $i$ .  $\square$

We will need, in particular, the following important consequence.

**Corollary 6.5.**  *$\mathcal{O}$  is invariant under  $\Lambda$  and  $H_{b_i} : \Lambda \mathcal{O} \subset \mathcal{O}$ ,  $H_{b_i} \mathcal{O} \subset \mathcal{O}$ . Similarly  $\mathcal{N}$  is invariant under  $\Lambda^*$  and  $H_{b_i} : \Lambda^* \mathcal{N} \subset \mathcal{N}$ ,  $H_{b_i} \mathcal{N} \subset \mathcal{N}$ .*

*Proof.* Immediate from the previous lemma.  $\square$

*Remark 6.6.* Given the previous corollary, it is not surprising that  $\mathcal{O}$  can in fact be characterized by its invariance property. To this end, denote by

$$\mathcal{O}_h = \{f \circ h : \forall f : \mathbb{H} \rightarrow \mathbb{R}\} = \text{span}\{H_{b_i} \mathbf{1} : b_i \in \mathbb{H}\}.$$

Then  $\mathcal{O}$  is the smallest subspace of  $\mathbb{R}^d$  that contains  $\mathcal{O}_h$  and is invariant under  $\Lambda$  and all  $H_{b_i}$ . Indeed, let us call this smallest subspace  $\mathcal{O}'$ . Clearly  $\mathcal{O}' \subset \mathcal{O}$ , as  $\mathcal{O}$  contains  $\mathcal{O}_h$  and is invariant under  $\Lambda$  and  $H_{b_i}$ . On the other hand, every element of  $\mathcal{O}$  can be generated from elements in  $\mathcal{O}_h$  by a finite number of multiplications by  $\Lambda$  and  $H_{b_i}$  and linear combinations. Hence  $\mathcal{O} \subset \mathcal{O}'$ .

To verify observability, we could proceed as follows. Denote

$$\mathcal{Z}_1 = \mathcal{O}_h, \quad \mathcal{Z}_n = \mathcal{Z}_{n-1} + \Lambda \mathcal{Z}_{n-1} + H_{b_1} \mathcal{Z}_{n-1} + \cdots + H_{b_r} \mathcal{Z}_{n-1}, \quad n > 1,$$

where the sum of two linear spaces denotes their linear span. It is evident that every element of  $\mathcal{O}$  will be in  $\mathcal{Z}_n$  for some  $n$ . Moreover, if  $\mathcal{Z}_n = \mathcal{Z}_{n+1}$  for some

$n = m$ , then it is true for all  $n > m$ , and in particular  $\mathcal{Z}_m = \mathcal{O}$ . Finally, we claim that this will always be the case for some  $m < d$ . Indeed, the dimension of  $\mathcal{Z}_n$  can not shrink with increasing  $n$ , but it can not grow larger than  $d$  as we are working in  $\mathbb{R}^d$ . As  $\mathcal{O}_h$  contains at least the constants, the procedure must complete in at most  $d - 1$  steps. This idea is classical, see, e.g., [3, section 3.2.2], and could be implemented, e.g., by starting with the natural basis  $\{H_{b_i} \mathbf{1}\}$  of  $\mathcal{O}_h$  and applying the Gram-Schmidt procedure at every iteration  $n$  to obtain a basis for  $\mathcal{Z}_n$ .

*Remark 6.7.* In an early paper on filter stability, Delyon and Zeitouni [13] impose a condition (A2) which, by the previous remark, is seen to be sufficient (but not necessary) for observability. In addition, they assume ergodicity of the signal process. Though their condition (A2) was later shown to be superfluous [4, theorem 4.1] in the nondegenerate case  $\kappa > 0$ , Delyon and Zeitouni show through a counterexample that when their condition (A2) is not satisfied, the filter may lose its stability as  $\kappa \rightarrow 0$ . That this can not happen when condition (A2) is satisfied is to be expected as, by corollary 4.6 above, observability implies filter stability without any nondegeneracy or ergodicity assumptions. It does not appear, however, that our results can be related to the methods used in [13], nor do our results give any information on the rate of convergence (exponential convergence is proved in [13]).

*Remark 6.8.* Denote by  $C = [H_{b_1} \mathbf{1}, \dots, H_{b_r} \mathbf{1}]$  the  $d \times r$  matrix whose columns are indicator functions on level sets of  $h$ . A sufficient (but not necessary) condition for observability is that  $\text{rank}([C \ \Lambda C \ \Lambda^2 C \ \dots \ \Lambda^{d-1} C]) = d$ , which is the classical observability test for linear systems. This corresponds to considering only the one dimensional distributions of  $h(X_t)$ , rather than all finite dimensional distributions.

**6.2. A complete characterization of filter stability when  $\kappa > 0$ .** Corollary 4.6 and the results of the previous section show that

**Corollary 6.9.** *If  $\dim \mathcal{O} = d$ , then the filter is stable.*

The converse, however, is not true. In the nondegenerate case  $\kappa > 0$ , it was shown by Baxendale, Chigansky and Liptser [4] that ergodicity of the signal is a sufficient condition for stability of the filter, regardless of the observation structure. It is not difficult to find an example of a filtering model that is not observable, but has an ergodic signal (e.g., choose any ergodic signal and set  $h = 0$ ; another example is the one in [13]).

The goal of this section is to find a necessary and sufficient condition for filter stability in the nondegenerate case  $\kappa > 0$ , which we will assume throughout. This is done by combining our results above with the results from [4]. To gain some intuition, recall that  $|\pi_t^\mu(f) - \pi_t^\nu(f)| \rightarrow 0$ , or, equivalently,  $|(\pi_t^\mu)^* f - (\pi_t^\nu)^* f| \rightarrow 0$ , for any  $f \in \mathcal{O}$ . This implies that as  $t \rightarrow \infty$ , the signed measure  $\pi_t^\mu - \pi_t^\nu$  converges to the nonobservable space  $\mathcal{N}$ . To ensure stability, we would like to find a condition under which the space  $\mathcal{N}$  converges to zero under the dynamics of the filter.

One plausible condition is to require that the signal itself “forgets” perturbations in  $\mathcal{N}$ , i.e., that  $|\mathbf{P}^\mu(X_t = a_i) - \mathbf{P}^\nu(X_t = a_i)| \rightarrow 0$  as  $t \rightarrow \infty$  for all  $a_i \in \mathbb{S}$  whenever  $\mu - \nu \in \mathcal{N}$ . In this way, we obtain the natural counterpart of the notion of *detectability* in linear systems theory. We will show that this condition is indeed necessary and sufficient for stability of the filter, provided that  $\kappa > 0$ .

Before turning to the proof, let us make precise what we are going to show. Recall that  $\Lambda^* \mathcal{N} \subset \mathcal{N}$ ; hence it makes sense to speak of the restriction  $\Lambda^*|_{\mathcal{N}}$ .



**Lemma 6.10.** Denote  $(p_t^\mu)^i = \mathbf{P}^\mu(X_t = a_i)$ , and suppose that we have  $\dim \mathcal{N} > 0$ . Then the following are equivalent statements.

- (1)  $|p_t^\mu - p_t^\nu| \rightarrow 0$  as  $t \rightarrow \infty$  whenever  $\mu - \nu \in \mathcal{N}$ .
- (2)  $\Lambda^*|_{\mathcal{N}}$  is Hurwitz (its eigenvalues have strictly negative real parts).
- (3)  $\Lambda^*|_{\mathcal{N}}$  has full rank.

Here  $|v|$  denotes the  $\ell_1$ -norm of the vector  $v$ .

*Proof.* The Kolmogorov forward equation states that

$$\frac{d}{dt} p_t^\mu = \Lambda^* p_t^\mu, \quad p_0^\mu = \mu.$$

Hence, in particular,

$$\frac{d}{dt} (p_t^\mu - p_t^\nu) = \Lambda^* (p_t^\mu - p_t^\nu) = \Lambda^*|_{\mathcal{N}} (p_t^\mu - p_t^\nu), \quad p_0^\mu - p_0^\nu = \mu - \nu \in \mathcal{N},$$

where the second equality follows immediately from the fact that this equation leaves  $\mathcal{N}$  invariant. It is well known from linear systems theory that the solution of this equation decays to zero as  $t \rightarrow \infty$  for every initial condition  $\mu - \nu \in \mathcal{N}$  if and only if  $\Lambda^*|_{\mathcal{N}}$  is Hurwitz. The fact that  $\Lambda^*|_{\mathcal{N}}$  is Hurwitz if and only if it is of full rank follows from the fact that any nonzero eigenvalue of  $\Lambda^*$  has strictly negative real part [2, pages 52–53].  $\square$

Our previous discussion now motivates the following definition.

**Definition 6.11.** The signal-observation model is called *detectable* if it is either observable or any of the equivalent conditions of lemma 6.10 hold.

The goal of this section is to prove the following theorem.

**Theorem 6.12.** Suppose that  $\kappa > 0$ . Then  $|\pi_t^\mu - \pi_t^\nu| \xrightarrow{t \rightarrow \infty} 0$   $\mathbf{P}^\mu$ -a.s. whenever  $\mu \ll \nu$  if and only if the signal-observation model is detectable.

*Remark 6.13.* The situation for  $\kappa = 0$  appears to be more complicated, and the theorem does not hold in this case. A counterexample can be found in [4, section 3] (see also [13]), which discusses a model that is certainly detectable, but the filter is not stable when  $\kappa = 0$  due to a sort of “geometric obstruction”. Problems of this sort, in somewhat different setting, date back to the work of Kaijser [18], and some recent progress on that problem can be found in [20]. A complete understanding of this case is still lacking, however.

**6.2.1. Necessity.** To prove theorem 6.12, we begin by showing that detectability is a necessary condition for the stability of the filter.

**Lemma 6.14.** Suppose that  $|\pi_t^\mu - \pi_t^\nu| \xrightarrow{t \rightarrow \infty} 0$   $\mathbf{P}^\mu$ -a.s. for any  $\mu \ll \nu$ . Then the signal-observation model is detectable.

*Proof.* Let  $\mu - \nu \in \mathcal{N}$ ; then  $\mathbf{P}^\mu$  and  $\mathbf{P}^\nu$  are identical on  $\mathcal{F}^Y$ . As  $\pi_t^\nu(f)$  is  $\mathcal{F}^Y$ -measurable,  $\mathbf{E}^\mu(\pi_t^\nu(f)) = \mathbf{E}^\nu(\pi_t^\nu(f))$ . Thus

$$\mathbf{E}^\mu |\pi_t^\mu(f) - \pi_t^\nu(f)| \geq |\mathbf{E}^\mu(\pi_t^\mu(f) - \pi_t^\nu(f))| = |\mathbf{E}^\mu(f(X_t)) - \mathbf{E}^\nu(f(X_t))| \quad \text{for any } f.$$

Now suppose the model is not detectable, i.e., there exists  $v \in \mathcal{N}$  so that  $|p_t^\mu - p_t^\nu| \not\rightarrow 0$  when  $\mu - \nu \propto v$ . Choose  $\nu = (v^+ + v^-)/2v^+(\mathbb{S})$  and  $\mu = v^+/v^+(\mathbb{S})$ ; then

$\mu, \nu \in \mathcal{P}(\mathbb{S})$ ,  $\mu \ll \nu$  and  $\mu - \nu \propto \nu$ . As  $p_t^\mu - p_t^\nu$  does not converge to zero, there must exist a function  $f \in \mathcal{C}_b(\mathbb{S})$  and a sequence of times  $t_n \nearrow \infty$  such that

$$|\mathbf{E}^\mu(f(X_{t_n})) - \mathbf{E}^\nu(f(X_{t_n}))| \xrightarrow{n \rightarrow \infty} \alpha > 0.$$

But if  $|\pi_t^\mu - \pi_t^\nu| \xrightarrow{t \rightarrow \infty} 0$   $\mathbf{P}^\mu$ -a.s., then using dominated convergence

$$|\mathbf{E}^\mu(f(X_{t_n})) - \mathbf{E}^\nu(f(X_{t_n}))| \leq \mathbf{E}^\mu|\pi_{t_n}^\mu(f) - \pi_{t_n}^\nu(f)| \xrightarrow{n \rightarrow \infty} 0.$$

Hence we have a contradiction, and the proof is complete.  $\square$

*Remark 6.15.* The previous proof does not use at all the fact that  $\mathbb{S}$  is a finite set or that the observations are of the white noise type. Indeed, let us call a general model *detectable* if  $\mu - \nu \in \mathcal{N}$  implies that  $|\mathbf{E}^\mu(f(X_t)) - \mathbf{E}^\nu(f(X_t))| \rightarrow 0$  as  $t \rightarrow \infty$  for any  $f \in \mathcal{C}_b(\mathbb{S})$ . Then precisely the same proof shows that detectability is a *necessary* condition for the stability of the filter (it is not even necessary to assume that  $\mathbb{S}$  is compact). The difficult part is to establish that detectability is a *sufficient* condition for stability of the filter, and this is what we will do below for finite state signals with nondegenerate white noise type observations.

**6.2.2. Sufficiency: no transient states.** We now proceed to prove that detectability is also a sufficient condition for stability when  $\kappa > 0$ . *Throughout this and the following subsection we always assume that the signal-observation model is detectable.*

In the proofs, we make use of the partition of the state space  $\mathbb{S}$  into  $M < \infty$  ergodic classes  $\mathbb{U}_i$ ,  $i = 1, \dots, M$  and a transient class  $\mathbb{T}$ , so that  $\mathbb{S}$  is the disjoint union of these sets. Any Markov chain can be uniquely decomposed in this way.

**Lemma 6.16.** *If  $\Lambda F = 0$  for a function  $F$ , then  $F \in \mathcal{O}$ .*

*Proof.*  $\Lambda F = 0$  is equivalent to  $e^{\Lambda t} F = F$  for all  $t \geq 0$ . In particular, this implies that  $(p_t^\mu)^* F = \mu^* F$  for all  $t \geq 0$ . Now suppose that  $F \notin \mathcal{O}$ ; then there exist  $\mu, \nu$  such that  $\mu - \nu \in \mathcal{N}$  and  $|\mu^* F - \nu^* F| = \alpha > 0$ . In particular, we find that  $|(p_t^\mu)^* F - (p_t^\nu)^* F| = \alpha$  for all  $t \geq 0$ . But detectability implies that  $|p_t^\mu - p_t^\nu| \rightarrow 0$  as  $t \rightarrow \infty$  for  $\mu - \nu \in \mathcal{N}$ , so we have a contradiction.  $\square$

**Corollary 6.17.** *Suppose  $\mathbb{T} = \emptyset$ . Then  $I_{\mathbb{U}_i} \in \mathcal{O}$  for any  $i = 1, \dots, M$ .*

*Proof.* It is easily seen that  $\Lambda I_{\mathbb{U}_i} = 0$  when there are no transient states. Hence the statement follows from the previous lemma.  $\square$

Suppose that  $\mathbb{T} = \emptyset$ . The essential consequence of detectability is that as  $t \rightarrow \infty$ , we will be able to determine precisely in which of the ergodic classes  $\mathbb{U}_1, \dots, \mathbb{U}_M$  the signal started at  $t = 0$ . Following the logic of [4], this will cause the filter to be stable when combined with the fact that the filter is stable for ergodic signals. We will deal with the transient states separately in the next subsection, and assume for now that there are no such states (or, equivalently, that we work with initial densities that are supported on the ergodic classes only).

**Lemma 6.18.**  *$\mathbf{E}^\nu(I_{\mathbb{U}_i}(X_0)|\mathcal{F}_t^Y) \xrightarrow{t \rightarrow \infty} I_{\mathbb{U}_i}(X_0)$   $\mathbf{P}^\nu$ -a.s., provided that there are no transient states  $\mathbb{T} = \emptyset$ .*

*Proof.* For any  $j$  such that  $\nu(\mathbb{U}_j) > 0$ , denote by  $\nu_j = I_{\mathbb{U}_j} \nu / \nu(\mathbb{U}_j)$ . Then  $\nu_j \ll \nu$  and  $\pi_t^{\nu_j}(I_{\mathbb{U}_i}) = \delta_{ij}$ . But by theorem 4.4  $|\pi_t^{\nu_j}(I_{\mathbb{U}_i}) - \pi_t^\nu(I_{\mathbb{U}_i})| \rightarrow 0$  as  $t \rightarrow \infty$   $\mathbf{P}^{\nu_j}$ -a.s., as  $I_{\mathbb{U}_i} \in \mathcal{O}$ . In other words,  $\mathbf{P}^\nu(\pi_t^\nu(I_{\mathbb{U}_i}) \not\rightarrow \delta_{ij} \text{ and } X_0 \in \mathbb{U}_j) = 0$ , so  $\pi_t^\nu(I_{\mathbb{U}_i}) \rightarrow \delta_{ij}$  on  $\{\omega : X_0 \in \mathbb{U}_j\}$ , modulo a  $\mathbf{P}^\nu$ -null set. Finally, note that  $I_{\mathbb{U}_i}(X_t) = I_{\mathbb{U}_i}(X_0)$   $\mathbf{P}^\nu$ -a.s., as the ergodic classes do not communicate.  $\square$

We can now prove sufficiency for the special case  $\mathbb{T} = \emptyset$ .

**Lemma 6.19.** *Suppose  $\mathbb{T} = \emptyset$  and  $\mu \ll \nu$ . Then  $|\pi_t^\mu(f) - \pi_t^\nu(f)| \rightarrow 0$   $\mathbf{P}^\mu$ -a.s. for all  $f \in \mathcal{C}_b(\mathbb{S})$ .*

*Proof.* By the Bayes formula, we find that  $\mathbf{P}^\mu$ -a.s.

$$\mathbf{E}^\mu(f(X_t)|\mathcal{F}_t^Y) = \sum_{j=1}^M \mathbf{E}^{\mu_j}(f(X_t)|\mathcal{F}_t^Y) \mathbf{P}^\mu(X_0 \in \mathbb{U}_j|\mathcal{F}_t^Y).$$

The same equation holds with  $\mu, \mu_j$  replaced by  $\nu, \nu_j$ . The result now follows easily from the previous lemma and the fact that  $|\mathbf{E}^{\mu_j}(f(X_t)|\mathcal{F}_t^Y) - \mathbf{E}^{\nu_j}(f(X_t)|\mathcal{F}_t^Y)| \rightarrow 0$  by [4, theorem 4.1] (as  $X_t$  is supported entirely in the ergodic class  $\mathbb{U}_j$  under the initial measures  $\mu_j, \nu_j$ ).  $\square$

**6.2.3. Sufficiency: general case.** We now consider the general case with  $\mathbb{T} \neq \emptyset$ . Let us begin by showing that the transient states themselves decay as  $t \rightarrow \infty$ .

**Lemma 6.20.**  $\pi_t^\nu(I_{\mathbb{T}}) \rightarrow 0$  as  $t \rightarrow \infty$   $\mathbf{P}^\nu$ -a.s.

*Proof.* Note that  $I_{\mathbb{T}}(X_t) \rightarrow 0$   $\mathbf{P}^\nu$ -a.s., as the transient states must decay eventually into one of the ergodic classes. Now write

$$\pi_t^\nu(I_{\mathbb{T}}) = \mathbf{E}^\nu(I_{\mathbb{T}}(X_t)|\mathcal{F}_t^Y) \leq \mathbf{E}^\nu\left(\sup_{u \geq n} I_{\mathbb{T}}(X_u) \middle| \mathcal{F}_t^Y\right)$$

for all  $t \geq n$   $\mathbf{P}^\nu$ -a.s. (using the càdlàg paths to eliminate the time dependence of the null set). Hence

$$\limsup_{t \rightarrow \infty} \mathbf{E}^\nu(I_{\mathbb{T}}(X_t)|\mathcal{F}_t^Y) \leq \mathbf{E}^\nu\left(\sup_{u \geq n} I_{\mathbb{T}}(X_u) \middle| \mathcal{F}^Y\right).$$

The claim follows by letting  $n \rightarrow \infty$  using dominated convergence.  $\square$

Evidently, as  $t \rightarrow \infty$ , the conditional measures  $\pi_t^\nu$  and  $\pi_t^\mu$  converge to measures that are supported on the ergodic classes  $\mathbb{U} = \mathbb{S} \setminus \mathbb{T} = \bigcup_{i=1}^M \mathbb{U}_i$ . On the other hand, if we start with  $\mu \ll \nu$  which are already supported on  $\mathbb{U}$ , then  $|\pi_t^\mu - \pi_t^\nu| \rightarrow 0$  by lemma 6.19. This strongly suggests that we should have  $|\pi_t^\mu - \pi_t^\nu| \rightarrow 0$  for any  $\mu \ll \nu$ . Our goal is to prove this assertion.

**Lemma 6.21.** *Suppose that  $\mu \ll \nu$  and that  $\mu$  is supported on  $\mathbb{U}$ . Then*

$$\mathbf{E}^\mu\left(\limsup_{t \rightarrow \infty} |\pi_t^\mu(f) - \pi_t^\nu(f)|\right) \leq \text{osc}(f) \|\mu/d\nu\|_\infty \nu(\mathbb{T}),$$

where  $\text{osc}(f) = \max(f) - \min(f)$ .

*Proof.* Let us write  $\nu_{\mathbb{U}} = I_{\mathbb{U}}\nu/\nu(\mathbb{U})$  and  $\nu_{\mathbb{T}} = I_{\mathbb{T}}\nu/\nu(\mathbb{T})$ . By the Bayes formula, we find that  $\mathbf{P}^\nu$ -a.s.

$$\pi_t^\nu(f) = \pi_t^{\nu_{\mathbb{U}}}(f) \mathbf{P}^\nu(X_0 \in \mathbb{U}|\mathcal{F}_t^Y) + \pi_t^{\nu_{\mathbb{T}}}(f) \mathbf{P}^\nu(X_0 \in \mathbb{T}|\mathcal{F}_t^Y).$$

It follows directly that  $\mathbf{P}^{\nu_{\mathbb{U}}}$ -a.s.

$$|\pi_t^{\nu_{\mathbb{U}}}(f) - \pi_t^\nu(f)| = |\pi_t^{\nu_{\mathbb{U}}}(f) - \pi_t^{\nu_{\mathbb{T}}}(f)| \mathbf{P}^\nu(X_0 \in \mathbb{T}|\mathcal{F}_t^Y),$$

so that in particular

$$\limsup_{t \rightarrow \infty} |\pi_t^{\nu_{\mathbb{U}}}(f) - \pi_t^\nu(f)| \leq \text{osc}(f) \mathbf{P}^\nu(X_0 \in \mathbb{T}|\mathcal{F}^Y).$$

We thus compute

$$\mathbf{E}^\mu(\mathbf{P}^\nu(X_0 \in \mathbb{T} | \mathcal{F}^Y)) \leq \|d\mu/d\nu\|_\infty \mathbf{E}^\nu(\mathbf{P}^\nu(X_0 \in \mathbb{T} | \mathcal{F}^Y)) = \|d\mu/d\nu\|_\infty \nu(\mathbb{T}).$$

The claim now follows from lemma 6.19 and

$$\limsup_{t \rightarrow \infty} |\pi_t^\mu(f) - \pi_t^\nu(f)| \leq \limsup_{t \rightarrow \infty} |\pi_t^\mu(f) - \pi_t^{\nu_U}(f)| + \limsup_{t \rightarrow \infty} |\pi_t^{\nu_U}(f) - \pi_t^\nu(f)|,$$

using the fact that  $\mu$  and  $\nu_U$  are both supported on  $\mathbb{U}$ .  $\square$

To establish that the right-hand side in the expression in this lemma can be chosen to be zero, we will use the Markov property of the filter.

**Lemma 6.22.** *For  $\mu \ll \nu$ , the pair  $(\pi_t^\mu, \pi_t^\nu)$  is a Feller-Markov process under  $\mathbf{P}^\mu$ .*

*Proof.* Recall that  $dB_t^\mu = \kappa^{-1}(dY_t - \pi_t^\mu(h) dt)$ , the innovations process, is an  $\mathcal{F}_t^Y$ -Wiener process under  $\mathbf{P}^\mu$ , and that we thus have

$$\begin{aligned} d\pi_t^\mu &= \Lambda^* \pi_t^\mu dt + \kappa^{-1} (H - h^* \pi_t^\mu) \pi_t^\mu dB_t^\mu, \\ d\pi_t^\nu &= \Lambda^* \pi_t^\nu dt + \kappa^{-1} (H - h^* \pi_t^\nu) \pi_t^\nu (dB_t^\mu + \kappa^{-1} h^* \pi_t^\mu dt - \kappa^{-1} h^* \pi_t^\nu dt), \end{aligned}$$

where  $H = \text{diag}(h)$  and  $(\pi_0^\mu, \pi_0^\nu) = (\mu, \nu)$ . For these facts, see, e.g., [25]. Being the solution of a stochastic differential equation with Lipschitz coefficients (the coefficients are bounded in the double simplex  $\Delta^d \times \Delta^d$ , and the first exit time from the simplex is infinite), it is well known that there is a unique strong solution which satisfies the Markov and Feller properties.  $\square$

A particular consequence of this lemma is the following. Consider the pair of  $\Delta^d$ -valued stochastic differential equations

$$\begin{aligned} d\pi_t &= \Lambda^* \pi_t dt + \kappa^{-1} (H - h^* \pi_t) \pi_t dW_t, \\ d\bar{\pi}_t &= \Lambda^* \bar{\pi}_t dt + \kappa^{-1} (H - h^* \bar{\pi}_t) \bar{\pi}_t (dW_t + \kappa^{-1} h^* \pi_t dt - \kappa^{-1} h^* \bar{\pi}_t dt), \end{aligned}$$

where  $W_t$  is a standard Wiener process. The solutions of this stochastic differential equation can be realized on the canonical path space  $\tilde{\Omega} = D([0, \infty]; \Delta^d) \times D([0, \infty]; \Delta^d)$  such that  $\pi_t(u, v) = u(t)$  and  $\bar{\pi}_t(u, v) = v(t)$  are the canonical processes, and with a family of measures  $\mathbb{P}_{(\mu, \nu)}$  under which  $(\pi_t, \bar{\pi}_t)$  solve the stochastic differential equation above for the initial condition  $(\pi_0, \bar{\pi}_0) = (\mu, \nu)$ . We can subsequently introduce the natural filtration  $\mathcal{E}_t = \sigma\{\pi_s, \bar{\pi}_s : s \leq t\}$ , augmented as usual with respect to the family  $\mathbb{P}_{(\mu, \nu)}$ , and the canonical shift  $\tilde{\theta}_t(u, v)(s) = (u(s+t), v(s+t))$ , such that the process  $(\pi_t, \bar{\pi}_t)$  satisfies the usual Markov property with respect to the filtration  $\mathcal{E}_t$  and the family  $\mathbb{P}_{(\mu, \nu)}$ . From the proof of the previous lemma, it follows that for any  $\mu \ll \nu$ , the law of the process  $(\pi_t, \bar{\pi}_t)$  under  $\mathbb{P}_{(\mu, \nu)}$  coincides with the law of the process  $(\pi_t^\mu, \pi_t^\nu)$  under  $\mathbf{P}^\mu$ . In particular, our previous results can be applied to the process  $(\pi_t, \bar{\pi}_t)$ , and to establish stability it suffices to demonstrate the corresponding property for the latter.

*Remark 6.23.* The construction of  $(\pi_t, \bar{\pi}_t)$  on its own path space is certainly not necessary, but helps alleviate some notational confusion. In particular, we will be using the Markov property of the filter, whereas our previous notation is geared at the Markov property of the signal-observation pair.

Combined with lemma 6.21, we can now establish the following.

**Lemma 6.24.** *Suppose that  $\mu \ll \nu$  and that  $\mu$  is supported on  $\mathbb{U}$ . Then it follows that  $|\pi_t^\mu(f) - \pi_t^\nu(f)| \rightarrow 0$   $\mathbf{P}^\mu$ -a.s. for any  $f \in \mathcal{C}_b(\mathbb{S})$ .*

*Proof.* Using the Markov property, we can write

$$\mathbb{E}_{(\mu, \nu)} \left( \limsup_{t \rightarrow \infty} |\pi_t(f) - \bar{\pi}_t(f)| \mid \mathcal{E}_s \right) = \mathbb{E}_{(\pi_s, \bar{\pi}_s)} \left( \limsup_{t \rightarrow \infty} |\pi_t(f) - \bar{\pi}_t(f)| \right),$$

where we have used the fact that the random variable  $\limsup_{t \rightarrow \infty} |\pi_t(f) - \bar{\pi}_t(f)|$  is invariant under the shift  $\bar{\theta}_s$ . By [9, lemma 2.1] we find that  $\pi_s \ll \bar{\pi}_s$   $\mathbb{P}_{(\mu, \nu)}$ -a.s. whenever  $\mu \ll \nu$ , whereas clearly  $\pi_s$  is  $\mathbb{P}_{(\mu, \nu)}$ -a.s. supported on  $\mathbb{U}$  whenever  $\mu$  is supported on  $\mathbb{U}$ . Hence we can invoke lemma 6.21, and we find that

$$\mathbb{E}_{(\mu, \nu)} \left( \limsup_{t \rightarrow \infty} |\pi_t(f) - \bar{\pi}_t(f)| \mid \mathcal{E}_s \right) \leq \text{osc}(f) \|d\pi_s/d\bar{\pi}_s\|_\infty \bar{\pi}_s(I_{\mathbb{T}}),$$

where  $\|d\pi_s/d\bar{\pi}_s\|_\infty = \max_i (d\pi_s/d\bar{\pi}_s)^i$ . In particular, this implies that

$$\mathbf{E}^\mu \left( \limsup_{t \rightarrow \infty} |\pi_t^\mu(f) - \pi_t^\nu(f)| \mid \mathcal{F}_s^Y \right) \leq \text{osc}(f) \|d\pi_s^\mu/d\pi_s^\nu\|_\infty \pi_s^\nu(I_{\mathbb{T}}).$$

Now note that (see, e.g., [9, lemma 2.1])

$$\left\| \frac{d\pi_s^\mu}{d\pi_s^\nu} \right\|_\infty = \max_i \frac{\mathbf{E}^\nu \left( \frac{d\mu}{d\nu}(X_0) \mid \mathcal{F}_s^Y, X_s = a_i \right)}{\mathbf{E}^\nu \left( \frac{d\mu}{d\nu}(X_0) \mid \mathcal{F}_s^Y \right)} \leq \frac{1}{\varrho_s} \left\| \frac{d\mu}{d\nu} \right\|_\infty.$$

Hence, letting  $s \rightarrow \infty$  and using lemma 6.20, we find that

$$\limsup_{t \rightarrow \infty} |\pi_t^\mu(f) - \pi_t^\nu(f)| = 0 \quad \text{on} \quad \{\omega : \varrho_\infty(\omega) > 0\} \setminus N,$$

where  $\mathbf{P}^\mu(N) = 0$ . But  $\mathbf{P}^\mu(\varrho_\infty > 0) = 1$ , so the result follows.  $\square$

We can now finally complete the proof. It is important to remember that we have assumed detectability throughout this subsection.

**Proposition 6.25.** *Suppose the signal-observation model is detectable. If  $\mu \ll \nu$ , then  $|\pi_t^\mu(f) - \pi_t^\nu(f)| \rightarrow 0$   $\mathbf{P}^\mu$ -a.s. for any  $f \in C_b(\mathbb{S})$ .*

*Proof.* By the previous lemma, we find that  $|\pi_t^{\mu\nu}(f) - \pi_t^\mu(f)| \rightarrow 0$  and  $|\pi_t^{\nu\nu}(f) - \pi_t^\nu(f)| \rightarrow 0$   $\mathbf{P}^{\mu\nu}$ -a.s. Hence, using the triangle inequality,  $|\pi_t^\mu(f) - \pi_t^\nu(f)| \rightarrow 0$   $\mathbf{P}^{\mu\nu}$ -a.s. But this implies, as in the proof of lemma 6.18, that  $|\pi_t^\mu(f) - \pi_t^\nu(f)| \rightarrow 0$  on  $\{\omega : X_0 \in \mathbb{U}\}$ , modulo a  $\mathbf{P}^\mu$ -null set. In particular, we can then estimate

$$\mathbf{E}^\mu \left( \limsup_{t \rightarrow \infty} |\pi_t^\mu(f) - \pi_t^\nu(f)| \right) \leq \text{osc}(f) \mu(\mathbb{T}).$$

To proceed, we apply the Markov property as in the previous proof. This yields

$$\mathbf{E}^\mu \left( \limsup_{t \rightarrow \infty} |\pi_t^\mu(f) - \pi_t^\nu(f)| \mid \mathcal{F}_s^Y \right) \leq \text{osc}(f) \pi_s^\mu(I_{\mathbb{T}}).$$

The result follows by letting  $s \rightarrow \infty$  and using lemma 6.20.  $\square$

## 7. STRONG STABILITY FOR NONDEGENERATE WHITE NOISE TYPE OBSERVATIONS

**7.1. Strong stability.** Up to this point, we have always assumed that the initial measures of interest are absolutely continuous  $\mu \ll \nu$ . In this section we consider the case when  $\mu \not\ll \nu$ . As explained in remark 4.8, the filter stability problem is in general not even well defined for such initial measures, and the characterization of strong stability (the stability of the filter for arbitrary initial conditions) requires choosing a particular version of the conditional expectations. In the case of nondegenerate white noise type observations, however, there is a natural choice

of version, viz. the one provided by the Kallianpur-Striebel formula, whose construction we now briefly recall (see, e.g., [25, section 7.9]).

We consider the generic white noise type observation model

$$Y_t = Y_0 + \int_0^t h(X_s) ds + KB_t$$

of section 5.1. Beside the assumptions of section 5.1, however, we additionally assume *nondegeneracy* of the observation process, i.e., we assume the  $K$  is an invertible matrix. The importance of this requirement stems from the fact that it allows us to use Girsanov's theorem to remove the dependence of the observations on the signal process, a fact which will be exploited shortly.

Denote by  $\mathbf{Q}^\mu$  the measure on the space of signal sample paths  $\Omega^X$  such that the canonical process  $X_t(x) = x(t)$  has the law of the signal process with initial distribution  $\mu$ , i.e.,  $\mathbf{Q}^\mu$  is the marginal of  $\mathbf{P}^\mu$  on  $\Omega^X$ . Moreover, we denote by  $\mathbf{W}$  the Wiener measure on  $\Omega^Y$  with covariance  $KK^*$  (i.e.,  $Y_t(y) = y(t)$  is a Wiener process with covariance  $KK^*$  under  $\mathbf{W}$ ). Then, by Girsanov's theorem,

$$Z_t = \frac{d\mathbf{P}^\mu|_{\mathcal{F}_t}}{d(\mathbf{Q}^\mu \times \mathbf{W})|_{\mathcal{F}_t}} = \exp \left( \int_0^t (KK^*)^{-1} h(X_s) \cdot dY_s - \frac{1}{2} \int_0^t \|K^{-1}h(X_s)\|^2 ds \right).$$

Thus, using the Bayes formula, we obtain the following characterization of the filter:

$$\pi_t^\mu(f)(x, y) = \mathbf{E}^\mu(f(X_t)|\mathcal{F}_t^Y)(x, y) = \frac{\int_{\Omega^X} Z_t(\tilde{x}, y) f(\tilde{x}(t)) \mathbf{Q}^\mu(d\tilde{x})}{\int_{\Omega^X} Z_t(\tilde{x}, y) \mathbf{Q}^\mu(d\tilde{x})} \quad \mathbf{P}^\mu\text{-a.s.}$$

This is the Kallianpur-Striebel formula. Note, however, that the expression on the right hand side depends only on the observation sample paths  $\Omega^Y$  in the time interval  $[0, t]$ , and is well defined not only  $\mathbf{P}^\mu$ -a.s. but in fact for  $\mathbf{W}$ -a.e.  $y \in \Omega^Y$ . Moreover, it is easily seen from Girsanov's theorem that the observation marginals satisfy  $\mathbf{P}^\mu|_{\mathcal{F}_T^Y} \sim \mathbf{W}|_{\mathcal{F}_T^Y}$  for any  $T < \infty$ , regardless of the initial measure  $\mu$ . Hence the Kallianpur-Striebel formula defines a version of  $\pi_t^\mu$  which is  $\mathbf{P}^\nu$ -a.s. uniquely defined for any  $\nu$  (even when  $\mu$  and  $\nu$  are not absolutely continuous). *In the remainder of this section,  $\pi_t^\mu$  will always imply this particular version.*

Having now chosen a version of the filter that is well defined under any measure  $\mathbf{P}^\mu$ , strong stability can be meaningfully defined.

**Definition 7.1.** The filtering model is *strong stable* if for any  $\mu, \nu, \gamma \in \mathcal{P}(\mathbb{S})$ ,

$$|\pi_t^\mu(f) - \pi_t^\nu(f)| \xrightarrow{t \rightarrow \infty} 0 \quad \mathbf{P}^\gamma\text{-a.s.} \quad \text{for all } f \in \mathcal{C}_b(\mathbb{S}).$$

**7.2. A complete characterization in the finite state case.** In the finite state setting, the condition  $\mu \ll \nu$  is not really restrictive in practice. Indeed, if we wish to ensure that the filter is asymptotically insensitive to its initial condition, it suffices to initialize the filter with a (possibly incorrect) initial distribution  $\nu$  which charges every point in the state space, e.g., the uniform distribution on  $\mathbb{S}$ . As any measure on  $\mathbb{S}$  is absolutely continuous with respect to such a measure, the convergence of the thus initialized filter to the optimal one is ensured, regardless of the true initial distribution  $\mu$ , provided that the model is detectable and  $\kappa > 0$ .

Nonetheless the strong stability property is of interest, and can be characterized completely as we did for the stability property. Somewhat surprisingly, the observation structure no longer plays a role in this setting.

**Theorem 7.2.** *Suppose that  $\kappa > 0$ . Then  $|\pi_t^\mu - \pi_t^\nu| \xrightarrow{t \rightarrow \infty} 0$   $\mathbf{P}^\gamma$ -a.s. for any  $\mu, \nu, \gamma$  if and only if the signal process has only one ergodic class.*

Before proceeding with the proof of the theorem, let us make the following important remark.

*Remark 7.3.* The stochastic differential equations used in the proof of lemma 6.22 can be obtained directly from the Kallianpur-Striebel formula, and hence define the version of  $(\pi_t^\mu, \pi_t^\nu)$  which we use in this section. In particular, this implies that the arguments based on the Markov property of  $(\pi_t^\mu, \pi_t^\nu)$  continue to hold in the current setting and the condition  $\mu \ll \nu$  of lemma 6.22 is no longer required.

Let us first prove the necessity part of the theorem. We assume throughout that  $\kappa > 0$  and that  $\pi_t^\mu, \pi_t^\nu$  are chosen to be the Kallianpur-Striebel versions.

**Lemma 7.4.** *Suppose that  $|\pi_t^\mu - \pi_t^\nu| \xrightarrow{t \rightarrow \infty} 0$   $\mathbf{P}^\gamma$ -a.s. for any  $\mu, \nu, \gamma$ . Then the signal process has only one ergodic class.*

*Proof.* Suppose that the signal process has two ergodic classes  $\mathbb{U}_1$  and  $\mathbb{U}_2$ . Choose  $\mu$  to be any distribution that is supported on  $\mathbb{U}_1$ , and  $\nu$  to be any distribution that is supported on  $\mathbb{U}_2$ . Then it is easily verified that  $\pi_t^\mu(I_{\mathbb{U}_1}) = 1$   $\mathbf{W}$ -a.s. for all times  $t$ , while  $\pi_t^\nu(I_{\mathbb{U}_1}) = 0$   $\mathbf{W}$ -a.s. for all times  $t$ . Hence we have a contradiction.  $\square$

We now proceed to prove sufficiency. First, note that it suffices to prove that  $|\pi_t^\mu - \pi_t^\nu| \rightarrow 0$   $\mathbf{P}^\mu$ -a.s. for any  $\mu, \nu$ . Indeed, it then follows that

$$|\pi_t^\mu - \pi_t^\nu| \leq |\pi_t^\mu - \pi_t^\gamma| + |\pi_t^\nu - \pi_t^\gamma| \xrightarrow{t \rightarrow \infty} 0 \quad \mathbf{P}^\gamma\text{-a.s.}$$

for any  $\mu, \nu, \gamma$  by the triangle inequality. As before, it is easier to first consider the case with no transient states  $\mathbb{T} = \emptyset$ .

**Lemma 7.5.** *Suppose the signal process is ergodic (in particular  $\mathbb{T} = \emptyset$ ). Then  $|\pi_t^\mu - \pi_t^\nu| \rightarrow 0$   $\mathbf{P}^\mu$ -a.s. for any  $\mu, \nu$ .*

*Proof.* This is precisely [4, theorem 4.1].  $\square$

Moreover, we need the following lemma.

**Lemma 7.6.** *Suppose the signal process has only one ergodic class  $\mathbb{U}$ . Then  $\pi_t^\mu(a_i) > 0$  a.s. for all  $a_i \in \mathbb{U}$  and  $t > 0$ , regardless of  $\mu$ .*

*Proof.* By [25, eq. (7.205)], we have  $\pi_t^\mu(a_i) > 0$  a.s. if and only if  $\mathbf{P}^\mu(X_t = a_i) > 0$ . But in the absence of multiple ergodic classes, it must be the case that  $\mathbf{P}^\mu(X_t = a_i) > 0$  for any  $a_i \in \mathbb{U}$  as soon as  $t > 0$ , regardless of  $\mu$ .  $\square$

The transient states can now be eliminated precisely as in proposition 6.25, completing the proof.

**Lemma 7.7.** *Suppose the signal process has only one ergodic class. Then we have  $|\pi_t^\mu - \pi_t^\nu| \rightarrow 0$   $\mathbf{P}^\mu$ -a.s. for any  $\mu, \nu$ .*

*Proof.* First, note that as in the proof of lemma 6.24, we can write

$$\mathbb{E}_{(\mu, \nu)} \left( \limsup_{t \rightarrow \infty} |\pi_t(f) - \bar{\pi}_t(f)| \mid \mathcal{E}_s \right) = \mathbb{E}_{(\pi_s, \bar{\pi}_s)} \left( \limsup_{t \rightarrow \infty} |\pi_t(f) - \bar{\pi}_t(f)| \right),$$

where we have used the Markov property and the fact that the random variable  $\limsup_{t \rightarrow \infty} |\pi_t(f) - \bar{\pi}_t(f)|$  is invariant under the shift  $\tilde{\theta}_s$ . But by the previous



lemma, we find that  $(\pi_t)_\mathbb{U} \sim (\bar{\pi}_t)_\mathbb{U}$  for any  $t > 0$ . Thus we may assume, without loss of generality, that  $\mu_\mathbb{U} \sim \nu_\mathbb{U}$  in the following.

By lemma 6.24, we find that  $|\pi_t^{\mu_\mathbb{U}}(f) - \pi_t^{\nu_\mathbb{U}}(f)| \rightarrow 0$   $\mathbf{P}^{\mu_\mathbb{U}}$ -a.s., and also that  $|\pi_t^{\nu_\mathbb{U}}(f) - \pi_t^{\nu'}(f)| \rightarrow 0$   $\mathbf{P}^{\nu_\mathbb{U}}$ -a.s. But from the assumption  $\mu_\mathbb{U} \sim \nu_\mathbb{U}$ , it follows that  $\mathbf{P}^{\mu_\mathbb{U}} \sim \mathbf{P}^{\nu_\mathbb{U}}$ . Hence, using the triangle inequality,  $|\pi_t^{\mu_\mathbb{U}}(f) - \pi_t^{\nu'}(f)| \rightarrow 0$   $\mathbf{P}^{\mu_\mathbb{U}}$ -a.s. But this implies, as in the proof of lemma 6.18, that  $|\pi_t^{\mu_\mathbb{U}}(f) - \pi_t^{\nu'}(f)| \rightarrow 0$  on  $\{\omega : X_0 \in \mathbb{U}\}$ , modulo a  $\mathbf{P}^\mu$ -null set. In particular, we can then estimate

$$\mathbf{E}^\mu \left( \limsup_{t \rightarrow \infty} |\pi_t^{\mu_\mathbb{U}}(f) - \pi_t^{\nu'}(f)| \right) \leq \text{osc}(f) \mu(\mathbb{T}).$$

To proceed, we apply the Markov property. This yields

$$\mathbf{E}^\mu \left( \limsup_{t \rightarrow \infty} |\pi_t^{\mu_\mathbb{U}}(f) - \pi_t^{\nu'}(f)| \mid \mathcal{F}_s^Y \right) \leq \text{osc}(f) \pi_s^\mu(I_\mathbb{T}).$$

The result follows by letting  $s \rightarrow \infty$  and using lemma 6.20.  $\square$

**7.3. A general criterion.** In this section, we will give a sufficient condition for strong stability for a general signal process (with nondegenerate white noise observations as above). We will need the following definitions.

**Definition 7.8.** Let  $R_t$  be a Markov semigroup on a Polish space with associated transition probabilities  $p_t(x, A)$ . Then  $R_t$  is called

- *strong Feller* if  $R_t f$  is continuous for every bounded measurable  $f$ ;
- *irreducible* if for any nonempty open set  $A$ , we have  $p_t(x, A) > 0$  for all  $x$ ;
- *regular* if  $p_t(x, \cdot) \sim p_t(y, \cdot)$  for every  $x, y$  and  $t > 0$ .

A well known result of Has'minskiĭ [16] states the following.

**Lemma 7.9.** *Any irreducible strong Feller semigroup is regular.*

The following theorem and its immediate corollary are the main results of this section. Recall that, by assumption, the signal is a Markov process in its own right.

**Theorem 7.10.** *If the signal process is regular, then stability implies strong stability.*

**Corollary 7.11.** *Regularity of the signal and observability imply strong stability of the filter. In particular, the filter is strong stable if the signal-observation model is observable and the signal process is irreducible and strong Feller.*

For the proof of the theorem, we need the following counterpart of lemma 6.22.

**Lemma 7.12.** *The pair  $(\pi_t^\mu, \pi_t^{\nu'})$  is a Feller-Markov process under  $\mathbf{P}^\mu$ .*

*Proof.* This follows as in the proof of [21, theorem 2.3]. The details are omitted.  $\square$

This implies that as in the finite state case, we can construct the filter on its canonical path space  $\tilde{\Omega} = D([0, \infty[; \mathcal{P}(\mathbb{S})) \times D([0, \infty[; \mathcal{P}(\mathbb{S}))$  (here  $\mathcal{P}(\mathbb{S})$  is endowed with the topology of weak convergence, which turns it into a compact Polish space). To be precise, denote by  $\mathbb{P}_{(\mu, \nu)}$  the probability measure on  $\tilde{\Omega}$  under which the canonical processes  $\pi_t(u, v) = u(t)$  and  $\bar{\pi}_t(u, v) = v(t)$  have the same law as do  $\pi_t^\mu$  and  $\pi_t^{\nu'}$  under  $\mathbf{P}^\mu$ . As before we introduce the natural filtration  $\mathcal{E}_t = \sigma\{(\pi_s, \bar{\pi}_s) : s \leq t\}$ , augmented with respect to the family  $\mathbb{P}_{(\mu, \nu)}$ , and the canonical shift  $\theta_t(u, v)(s) = (u(s+t), v(s+t))$ . It then follows that the process  $(\pi_t, \bar{\pi}_t)$  satisfies the usual Markov property with respect to the filtration  $\mathcal{E}_t$  and the family  $\mathbb{P}_{(\mu, \nu)}$ .

The strategy for proving theorem 7.10 is now straightforward. What we will show is that if the signal process is regular, then  $\pi_t^\mu \sim \pi_t^\nu$  a.s. for any  $t > 0$ , regardless of  $\mu$  and  $\nu$ . Using the Markov property of the filter, the strong stability problem then reduces to the ordinary stability problem.

*Proof of theorem 7.10.* Denote by  $\mathbf{Q}_t^\mu$  the law of  $X_t$  under  $\mathbf{P}^\mu$ . From regularity, it follows that  $\mathbf{Q}_t^\mu \sim \mathbf{Q}_t^\nu$  for any  $\mu, \nu$  and  $t > 0$ . But from the Kallianpur-Striebel formula, it follows directly that  $\pi_t^\mu \sim \mathbf{Q}_t^\mu$  a.s. with

$$\frac{d\pi_t^\mu}{d\mathbf{Q}_t^\mu}(z) = \frac{\int_{\Omega^X} Z_t(\tilde{x}, \cdot) \mathbf{Q}^\mu(d\tilde{x} | \tilde{x}(t) = z)}{\int_{\Omega^X} Z_t(\tilde{x}, \cdot) \mathbf{Q}^\mu(d\tilde{x})}.$$

Hence evidently  $\pi_t^\mu \sim \pi_t^\nu$  a.s. for any  $\mu, \nu$  and  $t > 0$ . Now note that for  $f \in \mathcal{C}_b(\mathbb{S})$ ,

$$\mathbb{E}_{(\mu, \nu)} \left( \limsup_{t \rightarrow \infty} |\pi_t(f) - \bar{\pi}_t(f)| \mid \mathcal{O}_s \right) = \mathbb{E}_{(\pi_s, \bar{\pi}_s)} \left( \limsup_{t \rightarrow \infty} |\pi_t(f) - \bar{\pi}_t(f)| \right),$$

where we have used the Markov property and the fact that the random variable  $\limsup_{t \rightarrow \infty} |\pi_t(f) - \bar{\pi}_t(f)|$  is invariant under the shift  $\tilde{\theta}_s$ . But we have just established that  $\pi_t \sim \bar{\pi}_t$  a.s., and thus the right hand side of this expression vanishes a.s. due to the fact that the filter is already assumed to be stable. Thus  $\limsup_{t \rightarrow \infty} |\pi_t(f) - \bar{\pi}_t(f)| = 0$  a.s., and the claim is established.  $\square$

The regularity of the signal process is closely related to the classical notion of controllability. Suppose that  $\mathbb{S}$  is a compact connected  $C^\infty$ -manifold, and that the signal process  $X_t$  is the solution of the Stratonovich stochastic differential equation

$$dX_t = F(X_t) dt + G(X_t) \circ dW_t, \quad X_0 \in \mathbb{S},$$

where  $F$  and  $G$  are  $C^\infty$ -vector fields on  $\mathbb{S}$ . Then  $X_t$  is a Markov process as usual with transition probabilities  $p_t(x, A)$ . Consider also the associated control system

$$\frac{d}{dt} \Xi_t = F(\Xi_t) + G(\Xi_t) u(t), \quad \Xi_0 \in \mathbb{S},$$

where  $u(t)$  is the control input. We denote by  $A_t(x) \subset \mathbb{S}$  the set of points  $\Xi_t$  which are reachable from  $\Xi_0 = x$  by the application of a piecewise smooth control signal  $u$ , and call the signal *controllable* if  $A_t(x) = \mathbb{S}$  for every  $x \in \mathbb{S}$  and  $t > 0$ . It follows from the Stroock-Varadhan support theorem that controllability is a sufficient condition for irreducibility of the signal [22]. Moreover, the controllability assumption additionally implies hypoellipticity of the diffusion  $X_t$  (see the remark on [22, page 175]), which gives rise to the strong Feller property.

Thus evidently a sufficient condition for strong stability of the filter, for a diffusion signal on a compact manifold with white noise type observations, is that the signal is controllable and the filtering model is observable. This mirrors precisely the well known controllability-observability criterion for the stability of the Kalman filter [6, 26]. Indeed, it is not difficult to verify that the linear filtering model is observable in the sense of this paper precisely when the well known observability rank condition is satisfied, while the linear signal is controllable precisely when the controllability rank condition is satisfied (though, unfortunately, the Kalman filter does not fit into the current setting as its state space is not compact).

*Remark 7.13.* Regularity of the signal process is certainly not a necessary condition for strong stability. In the finite state setting, for example, regularity occurs only when there is a single ergodic class and there are no transient states. We have seen,

however, that strong stability still holds true in the presence of transient states. The latter situation is analogous to the stabilizability criterion for the stability of the Kalman filter [26]. One might hope that also stabilizability and detectability have natural counterparts in the general setting, but we will not pursue this here.

On the other hand, we remark that stabilizability and detectability are generally considered together in the stability theory for the Kalman filter, while the results of this paper indicate that these conditions play rather separate roles. In particular, it is to be expected that detectability is a sufficient condition for the stability of the Kalman filter even in the absence of stabilizability, provided that the initial distributions are absolutely continuous  $\mu \ll \nu$ . That this is indeed the case (under slightly stronger conditions) is shown in the appendix.<sup>2</sup>

Both the Kalman filter and the finite state case give rise to conditions for observability and controllability which are easily computed explicitly in terms of matrices. For general diffusions, the matter appears to be much more complicated. To establish controllability one may employ certain Lie-algebraic computations, as detailed in [22]. The question of observability for signals on a non-finite state space does not appear to have been studied at all in the literature.

A slightly stronger condition than observability, however, is closely related to the classical observability problem for (deterministic) infinite-dimensional linear systems. Suppose that we have white noise type or counting observations, so that observability is determined by  $\mathcal{G}^h = \sigma\{h(X_t) : t \geq 0\}$ . Rather than require every  $\mu \neq \nu$  to merely give rise to different  $\mathbf{P}^\mu|_{\mathcal{G}^h} \neq \mathbf{P}^\nu|_{\mathcal{G}^h}$ , we could ask whether  $\mu \neq \nu$  implies that  $\mathbf{P}^\mu|_{\sigma\{h(X_t)\}} \neq \mathbf{P}^\nu|_{\sigma\{h(X_t)\}}$  for some  $t \geq 0$ . The latter is clearly a sufficient condition for observability, where only the one-dimensional distributions of the process  $h(X_t)$  are taken into account (compare with remark 6.8).

Now denote by  $R_t$  the Markov semigroup of the signal process. Then there exists a dual semigroup  $R_t^*$ , which acts on the space of measures  $\mathcal{M}(\mathbb{S})$ , such that  $R_t^*\mu$  is the law of  $X_t$  under  $\mathbf{P}^\mu$ . Moreover, let us define the projection map  $\Pi : \mathcal{M}(\mathbb{S}) \rightarrow \mathcal{M}(h(\mathbb{S}))$  such that  $\Pi : \mu \mapsto \mu \circ h^{-1}$ . Then we can consider  $\mathbf{X}_t = R_t^*\mu$  as defining the dynamics of an infinite-dimensional linear system with  $\mathbf{X}_0 = \mu$  and with infinite-dimensional linear observations  $\mathbf{Y}_t = \Pi\mathbf{X}_t$ . The classical observability problem associated with this infinite-dimensional linear model characterizes precisely when  $\mu \neq \nu$  implies that  $\mathbf{P}^\mu|_{\sigma\{h(X_t)\}} \neq \mathbf{P}^\nu|_{\sigma\{h(X_t)\}}$  for some  $t \geq 0$ . A detailed treatment of observability problems of this type can be found in [31].

## 8. THE NON-COMPACT CASE

In the preceding sections we have considered exclusively the case where the signal state space  $\mathbb{S}$  is compact, so that  $\mathcal{C}(\mathbb{S}) = \mathcal{C}_b(\mathbb{S}) = \mathcal{C}_0(\mathbb{S})$ . When  $\mathbb{S}$  is not compact, as in the common setting where  $\mathbb{S} = \mathbb{R}^n$ , for example, one could try to extend the proofs to show the stability of functions in  $\mathcal{C}_0(\mathbb{S})$ . The latter space of functions is the obvious choice from the point of view of our techniques, as  $\mathcal{C}_0(\mathbb{S})^* = \mathcal{M}(\mathbb{S})$  even when  $\mathbb{S}$  is only locally compact. However, from a practical point of view the stability of functions in  $\mathcal{C}_0(\mathbb{S})$  is too restrictive; indeed, if the signal is transient then the filtered estimate for any such function will be stable, but this fact is of little interest (as the filtered estimates of functions that vanish at infinity yield no

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<sup>2</sup> However, the method used in the appendix to prove stability of the Kalman filter is unrelated to the techniques developed in the body of this paper.

information on a transient signal as  $t \rightarrow \infty$ ). Instead, one should consider the larger class of continuous bounded functions  $\mathcal{C}_b(\mathbb{S})$  or of all continuous functions  $\mathcal{C}(\mathbb{S})$ .

Unfortunately, the techniques which we have developed in the previous sections do not extend directly to this setting. The problem is, of course, that the dual of  $\mathcal{C}_b(\mathbb{S})$  (with respect to the uniform topology) is no longer  $\mathcal{M}(\mathbb{S})$  when  $\mathbb{S}$  is not compact; rather,  $\mathcal{C}_b(\mathbb{S})^*$  can be characterized as  $\mathcal{M}(\beta\mathbb{S})$ , where  $\beta\mathbb{S}$  denotes the Stone-Ćech compactification of  $\mathbb{S}$ . A direct analog of our observability condition in this setting would thus require that no two initial measures give rise to the same observation statistics, even when those measures have some mass distribution “at infinity”. Though it is perhaps not surprising that the observability “at infinity” plays a role in this setting, the space  $\beta\mathbb{S}$  is sufficiently unwieldy that a direct extension of this type does not appear to lead to a useful theory.

In the remainder of this section we discuss two simple extensions of our results to the non-compact case. The first approach is inspired by the previous discussion; if the signal admits a *tractable* compactification  $\alpha\mathbb{S}$ , our previous results can be applied. This yields stability of those functions in  $\mathcal{C}_b(\mathbb{S})$  which admit a continuous extension to  $\alpha\mathbb{S}$ . The second approach assumes that the signal process is tight, so that the difficulties of a transient signal are avoided. In this case it is no longer necessary that  $\mathcal{O}$  is the uniform closure of  $\mathcal{O}^0$ ; using tightness, it is sufficient to consider the closure of  $\mathcal{O}^0$  with respect to the topology of uniform convergence on compact sets. This resolves our problems, as the dual of  $\mathcal{C}_b(\mathbb{S})$  endowed with the latter topology is  $\mathcal{M}_c(\mathbb{S})$ , the space of compactly supported finite signed measures.

*Remark 8.1.* It is only fair to remark that neither of these approaches is particularly satisfying. In particular, the natural test case for the theory, the Kalman filter with an unstable signal, is not covered by these approaches. (The Kalman filter for stable signals is not particularly interesting, as such filters are always stable regardless of observability; see, e.g., the result in the appendix). Further work is needed to develop an approach that covers unstable signals in a more satisfactory manner.

**8.1. Compactification.** We consider a signal-observation model  $(X_t, Y_t)$  as in section 2, except that  $\mathbb{S}$  is not assumed to be compact. Let us assume, furthermore, that the observations are of the white noise or counting type as in section 5, and that the observation function is continuous and bounded  $h \in \mathcal{C}_b(\mathbb{S})$ .

**Definition 8.2.** Let  $\alpha\mathbb{S}$  be a compact Polish space, and consider a filtering model  $(X_t^\alpha, Y_t^\alpha)$  with signal state space  $\alpha\mathbb{S}$  and observation function  $h^\alpha$  (the observation model is chosen to coincide with that of  $(X_t, Y_t)$ ). Then  $(X_t^\alpha, Y_t^\alpha)$  is a *compactification* of  $(X_t, Y_t)$  if there exists a continuous injection  $\pi : \mathbb{S} \rightarrow \alpha\mathbb{S}$  such that

- (1)  $h^\alpha \in \mathcal{C}_b(\alpha\mathbb{S})$  and  $h = h^\alpha \circ \pi$ ;
- (2) For any  $\mu \in \mathcal{P}(\mathbb{S})$ , the process  $(\pi(X_t), \pi(Y_t))$  with initial law  $(X_0, Y_0) \sim \mu$  has the same law as  $(X_t^\alpha, Y_t^\alpha)$  with initial law  $(X_0^\alpha, Y_0^\alpha) \sim \mu \circ \pi^{-1}$ .

The set  $\alpha\mathbb{S} \setminus \pi(\mathbb{S})$  is called the set of *points at infinity*.

Denote by  $\mathcal{C}_\alpha(\mathbb{S}) = \{f \in \mathcal{C}_b(\mathbb{S}) : f = f^\alpha \circ \pi \text{ for some } f^\alpha \in \mathcal{C}_b(\alpha\mathbb{S})\}$  the set of bounded continuous functions on  $\mathbb{S}$  that admit a continuous extension to  $\alpha\mathbb{S}$  (note that  $\mathcal{C}_\alpha(\mathbb{S})$  will always be strictly smaller than  $\mathcal{C}_b(\mathbb{S})$ , unless  $\alpha\mathbb{S} = \beta\mathbb{S}$ ). Then the following results follow immediately from our definitions.

**Proposition 8.3.** *If the filter for the model  $(X_t^\alpha, Y_t^\alpha)$  is stable, then the filter for the model  $(X_t, Y_t)$  is  $\alpha$ stable, i.e.,  $\mu \ll \nu$  implies*

$$|\mathbf{E}^\mu(f(X_t)|\mathcal{F}_t^Y) - \mathbf{E}^\nu(f(X_t)|\mathcal{F}_t^Y)| \rightarrow 0 \quad \mathbf{P}^\mu\text{-a.s.} \quad \text{for all } f \in \mathcal{C}_\alpha(\mathbb{S}).$$

**Corollary 8.4.** *Observability of the filtering model  $(X_t^\alpha, Y_t^\alpha)$  implies that the filter for  $(X_t, Y_t)$  is  $\alpha$ stable. In particular,  $\alpha$ stability is guaranteed if  $h^\alpha$  is one-to-one.*

We develop further a particular setting in which this result can be exploited. Set  $\mathbb{S} = \mathbb{R}^n$ , and consider a signal which solves the Itô stochastic differential equation

$$dX_t = f(X_t) dt + g(X_t) dW_t.$$

We assume that  $f, g$  are continuously differentiable and of sublinear growth:

$$f, g \in C^1, \quad \|f(x)\| \leq K(1 + \|x\|^\alpha), \quad \|g(x)\| \leq K(1 + \|x\|^\alpha), \quad \alpha < 1.$$

We now consider a compactification which adjoins to  $\mathbb{R}^n$  a sphere at infinity  $S^{n-1}$ ; this can be done, for example, by choosing  $\alpha\mathbb{S}$  to be the closed unit ball  $\{x \in \mathbb{R}^n : \|x\| \leq 1\}$  and setting  $\pi(x) = (1 + \|x\|^2)^{-1/2}x$ . E.g., when  $n = 1$ , this reduces to the two-point compactification  $[-\infty, \infty]$  of the real line  $\mathbb{R} = ]-\infty, \infty[$ , in which case  $\mathcal{C}_\alpha(\mathbb{R})$  is precisely the set of functions  $f \in \mathcal{C}_b(\mathbb{R})$  such that  $\lim_{x \rightarrow \pm\infty} f(x)$  exist.

Now choose  $h \in \mathcal{C}_\alpha(\mathbb{S})$ , and let  $Y_t$  be a white noise type or counting observation model with observation function  $h$ . Then it follows from [24, example 3] that there is a compactification  $(X_t^\alpha, Y_t^\alpha)$  of  $(X_t, Y_t)$  with the additional property that if  $X_0^\alpha \in \alpha\mathbb{S} \setminus \pi(\mathbb{S})$  a.s., then  $X_t^\alpha = X_0^\alpha$  for all  $t \geq 0$  a.s. (i.e., the points at infinity are fixed points for the compactified signal  $X_t^\alpha$ ). We can exploit the latter to give a criterion for  $\alpha$ stability in terms of properties of the non-compactified model.

**Proposition 8.5.** *Suppose that the following conditions hold:*

- (1)  $(X_t, Y_t)$  is observable (no two initial measures give rise to the same law of the observation process);
- (2) The restriction of  $h^\alpha$  to  $\alpha\mathbb{S} \setminus \pi(\mathbb{S})$  is one-to-one;
- (3)  $X_t$  does not possess an invariant manifold that is contained in some level set  $\{x \in \mathbb{S} : h(x) = u\}$  with  $u \in \{h^\alpha(x) : x \in \alpha\mathbb{S} \setminus \pi(\mathbb{S})\}$ .

*Then the filter for  $(X_t, Y_t)$  is  $\alpha$ stable.*

*Remark 8.6.* For condition (3) to be satisfied, it is sufficient to establish that we have  $\{h^\alpha(x) : x \in \alpha\mathbb{S} \setminus \pi(\mathbb{S})\} \cap \{h(x) : x \in \mathbb{S}\} = \emptyset$ . A sufficient condition for (1)–(3) to be satisfied is that  $h^\alpha$  is one-to-one.

*Proof.* Let  $\mu, \nu$  be two measures on  $\alpha\mathbb{S}$  with  $\mu \neq \nu$ . To prove the proposition, it suffices to establish that the law of  $h^\alpha(X_t^\alpha)$  is different for  $X_0^\alpha \sim \mu$  and  $X_0^\alpha \sim \nu$ .

First, suppose that  $\mu, \nu$  are supported on  $\alpha\mathbb{S} \setminus \pi(\mathbb{S})$ . Recall that if  $X_0^\alpha \in \alpha\mathbb{S} \setminus \pi(\mathbb{S})$ , then  $X_t^\alpha = X_0^\alpha$  for all  $t \geq 0$ . As  $h^\alpha$  is one-to-one on  $\alpha\mathbb{S} \setminus \pi(\mathbb{S})$ , this implies the claim.

Next, suppose that  $\mu$  is supported on  $\pi(\mathbb{S})$ , while  $\nu$  is supported on  $\alpha\mathbb{S} \setminus \pi(\mathbb{S})$ . If  $X_0^\alpha \sim \mu$  and  $X_0^\alpha \sim \nu$  were to give rise to the same law for  $h^\alpha(X_t^\alpha)$ , then  $h^\alpha(X_t^\alpha)$  would have to be an a.s. constant process under both measures, so that in particular  $\mu$  must be supported on the union of the invariant manifolds of  $X_t$  which are contained in level sets of  $h$ . But by the third assumption the process  $h^\alpha(X_t^\alpha)$  can then not take the same values under  $\mu$  and  $\nu$ , so that we have a contradiction.

Now let  $\mu$  and  $\nu$  be arbitrary. Then we can write  $\mu = a\mu_1 + (1 - a)\mu_2$  and  $\nu = b\nu_1 + (1 - b)\nu_2$ , where  $a, b \in [0, 1]$ ,  $\mu_1, \nu_1$  are supported on  $\pi(\mathbb{S})$ , and  $\mu_2, \nu_2$  are supported on  $\alpha\mathbb{S} \setminus \pi(\mathbb{S})$ . Note that by our assumptions, the probability that  $h^\alpha(X_t^\alpha)$

is a constant process which takes values in  $\{h^\alpha(x) : x \in \alpha\mathbb{S} \setminus \pi(\mathbb{S})\}$  is precisely  $1 - a$  when  $X_0^\alpha \sim \mu$  and  $1 - b$  when  $X_0^\alpha \sim \nu$ . Hence if  $\mu, \nu$  give rise to the same observation law, then  $a = b$  and  $\mu_2 = \nu_2$  (the latter follows as  $h^\alpha$  is assumed to be one-to-one on  $\alpha\mathbb{S} \setminus \pi(\mathbb{S})$ ). But then we can conclude that  $\mu, \nu$  give rise to the same observation law only if  $\mu_1$  and  $\nu_1$  give rise to the same observation law, and the latter implies  $\mu_1 = \nu_1$  by our first assumption. Hence the proof is complete.  $\square$

*Remark 8.7.* It is likely that the compactification approach described in this section can be generalized to a larger class of signals. However, the requirement that  $h \in \mathcal{C}_\alpha(\mathbb{S})$  appears to rule out any model in which the observation function is unbounded. In practice, on the other hand, the unbounded observation case is much more natural when  $\mathbb{S}$  is non-compact, and one would even expect stability to improve in this setting. The restriction to bounded observation functions is thus a significant drawback of the compactification approach. In particular, this rules out the application of this approach to the Kalman filter.

**8.2. Tight signal.** In this section, we consider a signal-observation model  $(X_t, Y_t)$  as in section 2, except that  $\mathbb{S}$  is assumed to be only locally compact. Throughout this subsection, the space of bounded continuous functions  $\mathcal{C}_b(\mathbb{S})$  will always be endowed with the topology of uniform convergence on compact sets. This is a locally convex topology, and gives rise to the duality  $\mathcal{C}_b(\mathbb{S})^* = \mathcal{M}_c(\mathbb{S})$  [10, proposition IV.4.1]. We note that lemma 3.5 extends also to this setting:

**Lemma 8.8.** *Let  $M \subset \mathcal{C}_b(\mathbb{S})$  be a linear subspace. Then  $(M^\perp)^\perp = \overline{M}$ , where  $\overline{M}$  is the closure of  $M$  in the topology of uniform convergence on compact sets.*

This can be proved in the same way as lemma 3.5, or follows as a special case of [10, theorem V.1.8]. In this setting, we will define

$$\begin{aligned} \mathcal{N} &= \{\alpha\mu_1 - \alpha\mu_2 \in \mathcal{M}_c(\mathbb{S}) : \alpha \in \mathbb{R}, \mu_1, \mu_2 \in \mathcal{P}_c(\mathbb{S}), \mu_1 \sim \mu_2\}, \\ \mathcal{O} &= \{f \in \mathcal{C}_b(\mathbb{S}) : \mu_1(f) = \mu_2(f) \text{ for all } \mu_1 \sim \mu_2 \text{ with } \mu_1, \mu_2 \in \mathcal{P}_c(\mathbb{S})\}. \end{aligned}$$

That is, we will consider bounded functions (which do not necessarily vanish at infinity), but we need only consider measures which are compactly supported. We now obtain the following analog of proposition 3.6.

**Proposition 8.9.** *Let  $\mathcal{O}^0$  be the linear span of functions of the form*

$$\mathbf{E}_{(x,y)}(f_1(Y_{t_1} - Y_0)f_2(Y_{t_2} - Y_0) \cdots f_n(Y_{t_n} - Y_0)),$$

*for all  $n < \infty$ ,  $t_i \in D$  and  $f_i \in \mathcal{C}_b(\mathbb{O})$ , where  $D$  is a dense subset of  $[0, \infty[$ . Then  $\mathcal{O}^0$  is dense in  $\mathcal{O}$  in the topology of uniform convergence on compact sets.*

The proof is identical to that of proposition 3.6, so we do not repeat it. We are now in the position to prove a stability theorem, similar to theorem 4.4, provided we assume that the signal is tight. The notion of stability is also somewhat weaker than that of theorem 4.4, as a.s. convergence is replaced by convergence in  $L^1$ .

**Theorem 8.10.** *Let  $\mu \ll \nu$ , and assume that the signal  $X_t$  is tight in the sense that for every  $\varepsilon > 0$ , there is a compact set  $K_\varepsilon \subset \mathbb{S}$  such that  $\mathbf{P}^\nu(X_t \in K_\varepsilon) > 1 - \varepsilon$  for all  $t \geq 0$ . Then  $\mathbf{E}^\mu(|\mathbf{E}^\mu(f(X_t)|\mathcal{F}_t^Y) - \mathbf{E}^\nu(f(X_t)|\mathcal{F}_t^Y)|) \xrightarrow{t \rightarrow \infty} 0$  for any  $f \in \mathcal{O}$ .*

*Proof.* For  $f \in \mathcal{O}^0$ , the result follows directly from lemma 4.1. Now choose  $\varepsilon, \delta > 0$ . By the tightness assumption and [32, page 146(b)], we can choose a compact set

$K$  such that  $\mathbf{P}^\mu(X_t \in K) > 1 - \varepsilon$  and  $\mathbf{P}^\nu(X_t \in K) > 1 - \varepsilon$  for all  $t \geq 0$ . We also choose  $f_\delta \in \mathcal{O}^0$  be such that  $\sup_{x \in K} |f(x) - f_\delta(x)| < \delta$ . Then

$$\begin{aligned} |\pi_t^\mu(f) - \pi_t^\nu(f)| &\leq |\pi_t^\mu(f) - \pi_t^\mu(fI_K)| + |\pi_t^\mu(fI_K) - \pi_t^\mu(f_\delta I_K)| \\ &\quad + |\pi_t^\mu(f_\delta I_K) - \pi_t^\mu(f_\delta)| + |\pi_t^\mu(f_\delta) - \pi_t^\nu(f_\delta)| + |\pi_t^\nu(f_\delta) - \pi_t^\nu(f_\delta I_K)| \\ &\quad + |\pi_t^\nu(f_\delta I_K) - \pi_t^\nu(fI_K)| + |\pi_t^\nu(fI_K) - \pi_t^\nu(f)|, \end{aligned}$$

where we have written  $\pi_t^\mu(f) = \mathbf{E}^\mu(f(X_t) | \mathcal{F}_t^Y)$ . Note that

$$\begin{aligned} \mathbf{E}^\mu(|\pi_t^\mu(f) - \pi_t^\mu(fI_K)|) &\leq \|f\| \mathbf{P}^\mu(X_t \in K^c) \leq \varepsilon \|f\|, \\ \mathbf{E}^\mu(|\pi_t^\mu(f_\delta I_K) - \pi_t^\mu(f_\delta)|) &\leq \|f_\delta\| \mathbf{P}^\mu(X_t \in K^c) \leq \varepsilon \|f_\delta\|, \end{aligned}$$

while

$$\mathbf{E}^\mu(|\pi_t^\mu(fI_K) - \pi_t^\mu(f_\delta I_K)|) \leq \delta, \quad \mathbf{E}^\mu(|\pi_t^\nu(f_\delta I_K) - \pi_t^\nu(fI_K)|) \leq \delta.$$

Now fix  $\gamma > 0$ , and note that

$$\begin{aligned} \mathbf{E}^\mu(|\pi_t^\nu(fI_K) - \pi_t^\nu(f)|) &\leq \|f\| \mathbf{E}^\nu\left(\frac{d\mu}{d\nu}(X_0) \pi_t^\nu(I_{K^c})\right) \\ &\leq \|f\| \mathbf{E}^\nu\left(\frac{d\mu}{d\nu}(X_0) I_{d\mu/d\nu(X_0) > \gamma}\right) + \gamma \|f\| \mathbf{P}^\nu(X_t \in K^c) \\ &\leq \|f\| \mathbf{E}^\nu\left(\frac{d\mu}{d\nu}(X_0) I_{d\mu/d\nu(X_0) > \gamma}\right) + \varepsilon \gamma \|f\|, \end{aligned}$$

and similarly

$$\mathbf{E}^\mu(|\pi_t^\nu(f_\delta) - \pi_t^\nu(f_\delta I_K)|) \leq \|f_\delta\| \mathbf{E}^\nu\left(\frac{d\mu}{d\nu}(X_0) I_{d\mu/d\nu(X_0) > \gamma}\right) + \varepsilon \gamma \|f_\delta\|.$$

It follows that

$$\begin{aligned} \limsup_{t \rightarrow \infty} \mathbf{E}^\mu(|\pi_t^\mu(f) - \pi_t^\nu(f)|) \\ \leq 2\delta + \left( \varepsilon + \varepsilon \gamma + \mathbf{E}^\nu\left(\frac{d\mu}{d\nu}(X_0) I_{d\mu/d\nu(X_0) > \gamma}\right) \right) (\|f\| + \|f_\delta\|). \end{aligned}$$

But  $\delta, \varepsilon, \gamma > 0$  were arbitrary, so the result follows by letting  $\delta, \varepsilon \rightarrow 0, \gamma \rightarrow \infty$ .  $\square$

*Remark 8.11.* If we consider all continuous functions  $\mathcal{C}(\mathbb{S})$ , rather than bounded continuous functions  $\mathcal{C}_b(\mathbb{S})$ , then it is still true that  $\mathcal{C}(\mathbb{S})^* = \mathcal{M}_c(\mathbb{S})$  when  $\mathcal{C}(\mathbb{S})$  is topologized by uniform convergence on compact sets. Thus, in fact, we may consider the set of continuous observable functions

$$\mathcal{O} = \{f \in \mathcal{C}(\mathbb{S}) : \mu_1(f) = \mu_2(f) \text{ for all } \mu_1 \sim \mu_2 \text{ with } \mu_1, \mu_2 \in \mathcal{P}_c(\mathbb{S})\},$$

and it is still the case that  $\mathcal{O}^0$  (which contains only bounded continuous functions) is dense in  $\mathcal{O}$ . This may be exploited, under suitable additional restrictions on the signal process, to prove stability of unbounded observable functions that satisfy an appropriate growth condition. For example, if  $\mathbb{S} = \mathbb{R}^n$ ,  $\|X_t\|^p$  is uniformly integrable and  $\|d\mu/d\nu\|_q < \infty$ , then the proof of the previous theorem is easily modified to show that observable functions of polynomial growth of degree  $k$  (depending on  $p, q$ ) are stable. See [8, proposition 3.3] for a similar argument.



## APPENDIX A. ON STABILITY OF THE KALMAN FILTER

We have shown, for a reasonably general class of nonlinear filters, that observability implies filter stability provided that we choose absolutely continuous initial measures  $\mu \ll \nu$ . A similar result for linear filtering models is well known, and the stability of the Kalman filter has been studied already for several decades (see, e.g., [6]; a more recent account can be found in [26]). These results, however, do not tend to assume that  $\mu \ll \nu$ , while on the other hand stabilizability is typically required in addition to detectability. The goal of this appendix is to illustrate that also in the linear setting, the stabilizability condition can be disposed of if we are willing to impose an absolute continuity requirement on the initial conditions. This highlights the separate roles of stabilizability and detectability in this setting.

*Remark A.1.* We will make no attempt at generality, and prove only the simplest possible result, for the purpose of illustration, by applying readily available results from the literature. Despite that the conclusion is hardly surprising and that the proof is straightforward, the author could not find any such result in the literature.

We consider the following linear signal-observation model:

$$\begin{aligned} dX_t &= AX_t dt + B dW_t, \\ dY_t &= CX_t dt + dB_t, \end{aligned}$$

where  $A$ ,  $B$ , and  $C$  are  $n \times n$ ,  $n \times m$ , and  $p \times n$  matrices, respectively, and  $X_0$  is Gaussian with mean  $\hat{X}_0$  and covariance matrix  $P_0$ . As is well known, the filtered estimate  $\hat{X}_t = \mathbf{E}(X_t | \mathcal{F}_t^Y)$  and covariance  $P_t = \mathbf{E}((X_t - \hat{X}_t)(X_t - \hat{X}_t)^*)$  satisfy

$$\begin{aligned} d\hat{X}_t &= A\hat{X}_t dt + P_t C^* (dY_t - C\hat{X}_t dt), \\ \frac{dP_t}{dt} &= AP_t + P_t A^* + BB^* - P_t C^* C P_t. \end{aligned}$$

The first equation is the Kalman filtering equation, while the second is the Riccati equation. We would like to compare the solution of these equations with the solutions  $\hat{X}'_t$ ,  $P'_t$  of the same equations with incorrect initial conditions  $\hat{X}'_0$ ,  $P'_0$ .

**Proposition A.2.** *Suppose  $(A, C)$  is detectable and  $P_0 > 0$ ,  $P'_0 > 0$ . Then there is a  $P_\infty$  such that  $P_t, P'_t \rightarrow P_\infty$  as  $t \rightarrow \infty$ , and  $\mathbf{E}\|\hat{X}_t - \hat{X}'_t\| \rightarrow 0$  as  $t \rightarrow \infty$ .*

*Remark A.3.* It is known that under the conditions of the proposition the solution of the Riccati equation converges to a unique limit  $P_\infty$  [11]. There is a key difference with the stabilizable case, however: in the current setting, the matrix  $A - P_\infty C^* C$  could be singular, while in the stabilizable case the matrix  $A - P_\infty C^* C$  is guaranteed to be strictly negative. The current situation is thus more subtle, and the proof of [26] does not immediately extend to this setting.

*Proof.* The existence of the (nonnegative definite) matrix  $P_\infty$  and the convergence of  $P_t, P'_t$  is established in [11] (see also [27] for more recent results). It remains to establish the second part of the proposition. To this end, recall that the innovation  $d\bar{B}_t = dY_t - C\hat{X}_t dt$  is a Wiener process. We begin by writing

$$d(\hat{X}_t - \hat{X}'_t) = (A - P'_t C^* C)(\hat{X}_t - \hat{X}'_t) dt + (P_t - P'_t) C^* d\bar{B}_t.$$

Now recall that the fact that  $(A, C)$  is detectable implies that there exists a matrix  $K$  such that  $A - KC$  only has eigenvalues with strictly negative real parts. Fix any

such  $K$ , define  $F = KC - A$ , and note that we can write

$$\hat{X}_T - \hat{X}'_T = e^{-FT}(\hat{X}_0 - \hat{X}'_0) + \int_0^T e^{-F(T-t)}(K - P'_t C^*)C(\hat{X}_t - \hat{X}'_t) dt + \int_0^T e^{-F(T-t)}(P_t - P'_t)C^* d\bar{B}_t.$$

We claim that each of these terms converges to zero in  $L^1$ . Let us consider each term individually. The first term clearly converges to zero, as the eigenvalues of  $F$  have strictly positive real parts. For the second term, note that there are constants  $c_1, \lambda > 0$  such that  $\|e^{-F(T-t)}\| \leq c_1 e^{-\lambda(T-t)}$ , and that as  $P'_t \rightarrow P_\infty$  it must be the case that  $\|P'_t\|$  is bounded from above by some constant  $c_2$ . We can thus estimate

$$\left\| \int_0^T e^{-F(T-t)}(K - P'_t C^*)C(\hat{X}_t - \hat{X}'_t) dt \right\| \leq c_3 \int_0^T e^{-\lambda(T-t)} \|C(\hat{X}_t - \hat{X}'_t)\| dt,$$

where we have lumped the various constants into  $c_3 > 0$ . But by [9, theorem 3.1]

$$\mathbf{E} \left[ \int_0^\infty \|C(\hat{X}_t - \hat{X}'_t)\|^2 dt \right] < \infty \implies \int_0^\infty (\mathbf{E} \|C(\hat{X}_t - \hat{X}'_t)\|)^2 dt < \infty.$$

Hence the second term converges to zero in  $L^1$  as  $T \rightarrow \infty$  by lemma A.4 below. It remains to deal with the third term. To this end, note that

$$\mathbf{E} \left( \left\| \int_0^T e^{-F(T-t)}(P_t - P'_t)C^* d\bar{B}_t \right\|^2 \right) = \int_0^T \|e^{-F(T-t)}(P_t - P'_t)C^*\|^2 dt,$$

where  $\|\cdot\|$  is the Frobenius norm. It is easily seen, again using lemma A.4 below, that this expression converges to zero as  $T \rightarrow \infty$ . Hence the stochastic integral converges to zero in  $L^2$ , and thus also in  $L^1$ . The proof is complete.  $\square$

The following simple lemma was used in the proof.

**Lemma A.4.** *Let  $\lambda > 0$  and  $f : [0, \infty[ \rightarrow \mathbb{R}$ . Then*

$$\int_0^\infty f(t)^2 dt < \infty \implies \int_0^T e^{-\lambda(T-t)} f(t) dt \xrightarrow{T \rightarrow \infty} 0.$$

*The result also holds if we require  $f(t) \rightarrow 0$  instead of the  $L^2$  bound.*

*Proof.* Note that for any  $s > 0$ , we can estimate

$$\begin{aligned} \limsup_{T \rightarrow \infty} \left| \int_0^T e^{-\lambda(T-t)} f(t) dt \right| \\ \leq \limsup_{T \rightarrow \infty} \left| \int_0^s e^{-\lambda(T-t)} f(t) dt \right| + \limsup_{T \rightarrow \infty} \left| \int_s^T e^{-\lambda(T-t)} f(t) dt \right|. \end{aligned}$$

Clearly the first term on the right is zero. For the second term, note that

$$\left| \int_s^T e^{-\lambda(T-t)} f(t) dt \right| \leq \left[ \int_s^T e^{-2\lambda(T-t)} dt \int_s^T f(t)^2 dt \right]^{1/2},$$

by Cauchy-Schwarz. The result follows by letting  $T \rightarrow \infty$ , then  $s \rightarrow \infty$ . To prove that the result also holds if  $f(t) \rightarrow 0$ , it suffices to repeat the proof with

$$\left| \int_s^T e^{-\lambda(T-t)} f(t) dt \right| \leq \left[ \sup_{t \geq s} |f(t)| \right] \int_s^T e^{-\lambda(T-t)} dt,$$

where it should be noted that  $f(t) \rightarrow 0$  implies  $\limsup_{t \rightarrow \infty} |f(t)| = 0$ .  $\square$

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